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# The AMERICAN MATHEMATICAL MONTHLY

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# THE CIRCLES OF CURVATURE OF THE CURVES OF STEEPEST DESCENT OF GREEN'S FUNCTION*

J. L. WALSH, Harvard University

A number of years ago the present writer published [1] a study of the tangents and circles of curvature of lemniscates and of the level loci of Green's function in the plane. The present note is complementary to that previous study, and considers elementary geometric properties of the curves of steepest descent, namely the lines of force, or orthogonal trajectories of the level loci of Green's function. The method used is a continuation of the previous methods, especially use of an integral representation of Green's function essentially due to Hilbert.

The previous paper pointed out that for an infinite plane region $R$ with finite boundary $B$, the normal to the level locus of Green's function $G(x, y)$ with pole at infinity, at an arbitrary point $P$ of $R$, cuts the smallest convex point set $K$ containing $B$. That normal is, of course, the tangent at $P$ to the curve of steepest descent passing through $P$, and the tangent thus cuts $K$. All critical points of $G(x, y)$ in $R$ lie in $K$, and these are the only multiple points in $R$ of the level loci or of the curves of steepest descent; at every finite point of $R$ exterior to $K$ the tangents and normals to these curves exist and are unique.

The principal result of this note is

THEOREM 1. *Let $G(x, y)$ be Green's function with pole at infinity for an infinite region $R$ whose boundary $B$ is finite. Let a point $P$ of $R$ lie exterior to a circle $\Gamma$ containing $B$, and let the tangent at $P$ to the curve of steepest descent of $G(x, y)$ through $P$ pass through the center of $\Gamma$. Then the circle of curvature of that curve at $P$ cuts $\Gamma$.*

It is sufficient here to prove the conclusion in the case that $B$ consists wholly of a finite number of mutually exterior analytic Jordan curves, for if $B$ is replaced by a locus $B_1$: $G(x, y) = A$ (const.) in $R$, Green's function with pole at infinity for the infinite region $R_1$ bounded by $B_1$ is $G_1(x, y) \equiv G(x, y) - A$; the curves of steepest descent of $G_1(x, y)$ in $R_1$ are precisely those of $G(x, y)$. The locus $G(x, y) = A$ in $R$ consists of a finite number of mutually exterior analytic Jordan curves for all but a countable set of values of $A (> 0)$, and this locus can be chosen as near $B$ as desired. Indeed, we consider an infinite sequence of such loci $B_k$: $G(x, y) = A_k$ in $R$, where $A_k$ approaches zero monotonically. If the circle of curvature in Theorem 1 cuts each of a sequence of concentric circles $\Gamma_k$ containing $B_k$ with monotonically decreasing radii, that circle of curvature also cuts the limiting circumference $\Gamma$.

Under the assumption that $B$ consists of a finite number of mutually exterior analytic Jordan curves, it can be shown ([1]; [2], Sec. 4.2) that $G(x, y)$ admits a representation for $(x, y)$ in $R$ of the form

* This research was sponsored in part by the U. S. Air Force, Office of Scientific Research of the Air Research and Development Command.

$$G(x, y) = \int_B \log r d\sigma + g, \qquad r^2 = (x - \alpha)^2 + (y - \beta)^2, \tag{1}$$

where $(\alpha, \beta)$ are the running coordinates, $\sigma$ is a suitably chosen distribution on $B$, and $g$ is a constant. At the point $P$: $(0, b)$, $b<0$, of $R$ the partial derivatives of $G(x, y)$ are given by the following formulas, where all integrals are taken over $B$:

$$\begin{aligned} G_x &= \int -\alpha r^{-2} d\sigma, & G_y &= \int (b - \beta) r^{-2} d\sigma, \\ G_{xx} &= \int \{(b - \beta)^2 - \alpha^2\} r^{-4} d\sigma, & G_{xy} &= \int 2\alpha(b - \beta) r^{-4} d\sigma. \end{aligned} \tag{2}$$

Green's function $G(x, y)$ is harmonic in $R$ except at infinity, and possesses there a conjugate function $H(x, y)$, also harmonic but not necessarily single valued in $R$. The level loci of $H(x, y)$ in $R$ are precisely the curves of steepest descent of $G(x, y)$. To determine their circles of curvature we shall need the partial derivatives of $H(x, y)$, of which those of first order are given by the Cauchy-Riemann equations:

$$G_x = H_y, \quad G_y = -H_x, \quad G_{xx} = H_{xy}, \quad G_{xy} = H_{yy} = -H_{xx}, \quad G_{yy} = -H_{xy}. \tag{3}$$

To prove Theorem 1 we choose $\Gamma$ as $x^2+y^2=a^2$, so we have

$$\alpha^2 + \beta^2 \leqq a. \tag{4}$$

Moreover the slope $-H_x/H_y$ of the level locus of $H(x, y)$ through $P$ is infinite at $P$, whence

$$H_y = G_x = \int -\alpha r^{-2} d\sigma = 0. \tag{5}$$

It is to be noted that $P$ is not a critical point of $G(x, y)$ and $H(x, y)$, so we have there $H_x = -G_y \neq 0$. The center of curvature $C$: $(X, Y)$ of the level locus of $H(x, y)$ at $P$ has the coordinates $X = \pm\rho$, $Y = b$, where $\rho(\geqq 0)$ is the radius of curvature (compare [1]) at $P$:

$$\rho = \pm \frac{[H_x^2 + H_y^2]^{3/2}}{H_y^2 H_{xx} - 2H_x H_y H_{xy} + H_x^2 H_{yy}} = \pm \frac{H_x}{H_{yy}},$$

$$\rho = \mp \frac{\int (b - \beta) r^{-2} d\sigma}{\int 2\alpha(b - \beta) r^{-4} d\sigma}. \tag{6}$$

Since the circle of curvature is tangent to $OP$ at $P$, the condition that that circle should cut $\Gamma$ can be written

$$X^2 + b^2 \leqq (\rho + a)^2, \tag{7}$$

$$b^2 - a^2 \leqq 2a\rho. \tag{8}$$

We note that $b-\beta<0$, since $P$ is exterior to $\Gamma$, so (8) can be written

$$(b^2 - a^2)\int \alpha(b - \beta)r^{-4}d\sigma \leqq -\int a(b - \beta)r^{-2}d\sigma, \tag{9}$$

provided we have

$$\int \alpha(b - \beta)r^{-4}d\sigma > 0. \tag{10}$$

If the first member of (10) is zero, we have $\rho=\infty$, the circle of curvature coincides with the straight line $OP$, and the conclusion of the theorem is satisfied. If the first member of (10) is negative, reflection of the region $R$ in the $y$-axis makes no essential change in hypothesis or conclusion, and replaces $x$ by $-x$, $\alpha$ and $-\alpha$, and achieves (10). Thus it remains to study (9) and (10).

We rewrite (9) as

$$0 \leqq \int \frac{(b - \beta)[-\alpha(b^2 - a^2) - ar^2]}{r^4}\, d\sigma. \tag{11}$$

By the Cauchy inequality in the form $|A\alpha+B\beta| \leqq (A^2+B^2)^{1/2}(\alpha^2+\beta^2)^{1/2}$, by $r^2=\alpha^2+(b-\beta)^2$, and by (4), we may write

$$\begin{aligned}
-\alpha(b^2 - a^2) - ar^2 &= -ab^2 - a(\alpha^2 + \beta^2) - (b^2 - a^2)\alpha + 2ab\beta \\
&\leqq -ab^2 - a(\alpha^2 + \beta^2) + [(b^2 - a^2)^2 + (2ab)^2]^{1/2}(\alpha^2 + \beta^2)^{1/2} \\
&= -ab^2 - a(\alpha^2 + \beta^2) + (a^2 + b^2)(\alpha^2 + \beta^2)^{1/2} \\
&= [b^2 - a(\alpha^2 + \beta^2)^{1/2}][-a + (\alpha^2 + \beta^2)^{1/2}] \leqq 0.
\end{aligned}$$

Thus the integrand in (11) is nonnegative, so (8) is established and if $B$ lies interior to $\Gamma$ is established even with the strong inequality.

A limiting case of Theorem 1 is easily proved:

THEOREM 2. *Let $G(x, y)$ be Green's function with pole at infinity for an infinite region $R$ whose boundary $B$ is finite. Let a point $P$ of $R$ lie exterior to a half-plane (bounded by a line $L$) containing $B$, and let the tangent at $P$ to the curve of steepest descent through $P$ be perpendicular to $L$. Then the circle of curvature of that curve at $P$ cuts $L$.*

Theorem 2 is readily proved from Theorem 1 by noting that the circle of curvature cuts all circles $\Gamma$ containing $B$ but not containing $P$ whose centers lie on the perpendicular to $L$ through $P$. Consequently the circle of curvature

also cuts $L$. A proof of Theorem 2 can also be given by the methods previously used. Let $P$ be $(0, b)$, $b<0$, and let $L$ be $Ox$. Then $B$ lies in the half-plane $y \geqq 0$. With the assumption (10) as before, (6) with the upper sign is valid. We are to prove $\rho > -b$, namely,

$$-\int (b-\beta) r^{-2} d\sigma > \int -2\alpha b (b-\beta) r^{-4} d\sigma,$$

$$-\int (b-\beta)[r^2 - 2\alpha b] r^{-4} d\sigma > 0. \tag{12}$$

However, we have $r^2 = \alpha^2 + (b-\beta)^2 \geqq \alpha^2 + b^2 \geqq 2|\alpha b|$. The inequality $r^2 - 2\alpha b \geqq 0$ becomes an equality only in the point $\alpha = b, \beta = 0$; by (5) the set $B$ contains other points, so (12) follows and thereby Theorem 2.

Theorem 1 can be expressed in a form invariant under linear transformation of the complex variable; such a transformation of the extended plane carries every "circle" (where the term is used to include straight line) into a "circle," and also transforms Green's function for a region into Green's function for the transformed region. Thus we have

THEOREM 3. *Let $R$ be a region of the extended plane and $G(x, y)$ Green's function for $R$ with pole in some point $Q$ of $R$. Let $\Lambda$ be the circle of curvature at the point $P$ in $R$ to the curve of steepest descent for $G(x, y)$ through $P$. Let the circle tangent to this curve at $P$ and passing through $Q$ be orthogonal to the boundary $\Gamma$ of a circular region (closed interior of a circle, closed exterior of a circle, or closed half-plane) not containing $P$ or $Q$ which contains the boundary $B$ of $R$. Then $\Lambda$ cuts $\Gamma$.*

A limiting case of Theorem 3 is the invariant form of Theorem 2, which can be easily formulated by the reader. If $R$ is simply connected, the curves of steepest descent of $G(x, y)$ are of course the images of the radii (*Radienbilder*) when the unit circle is mapped conformally and one-to-one onto $R$ so that the center of the circle is transformed into $Q$.

If $R$ is a region which does not possess a Green's function in the classical sense, it may still possess in an extended sense a Green's function $G(x, y)$ ($\not\equiv +\infty$), found as the limit of the sequence of classical Green's functions $G_k(x, y)$ all with fixed pole $Q$ for a sequence $R_k$ of subregions of $R$ which monotonically increase and exhaust $R$ (compare [1], Sec. 15). Theorem 3 remains valid for $G(x, y)$ in this case. In fact, the functions $G_k(x, y)$ approach $G(x, y)$ uniformly in any closed bounded subset of $R$, and the partial derivatives of the $G_k(x, y)$ approach uniformly the corresponding partial derivatives of $G(x, y)$ in such a subset. Thus the tangents and circles of curvature of the curves of steepest descent of the $G_k(x, y)$ approach the tangents and circles of curvature of $G(x, y)$. A suitably chosen sequence of circular regions bounded by "circles" $\Gamma_k$ containing the respective boundaries of the $R_k$ approaches the circular region bounded by $\Gamma$ containing $B$, where each $\Gamma_k$ is normal to the tangent at $P$ to the curve of steepest

descent through $P$ for $G_k(x, y)$. The circles of curvature $\Lambda_k$ of these curves at $P$ cut the respective circles $\Gamma_k$ by Theorem 3, so the limit $\Lambda$ of the $\Lambda_k$ cuts the limit $\Gamma$ of the $\Gamma_k$.

It is pointed out in [1] that the technique used there applies to the study of curvature of both lemniscates and level loci of Green's function. A lemniscate is defined as a locus

$$|p(z)| = \text{const.}, \qquad p(z) \equiv \prod_1^n (z - z_k), \tag{13}$$

and of course the lemniscate can be defined equivalently as the locus

$$\sum_1^n \log |z - z_k| = A, \qquad A = \text{const.} \tag{14}$$

The first member of (14) is closely analogous to the second member of (1); in (14) there appears a finite sum and in (1) an integral (plus a constant which is here without significance). Results concerning lemniscates can be proved (i) by using a succession of constants $A_k$ in (14), $A_k \to -\infty$, and interpreting $(1/n)[\sum_1^n \log |z-z_k| - A_k]$ as a Green's function or (ii) by using discussions directly involving finite sums rather than integrals. By either method there may be proved

THEOREM 4. *Let $p(z) \equiv \prod_1^n (z-z_k)$ be a polynomial, let a point $P$ lie exterior to a circle $\Gamma$ containing the points $z_k$, and let the tangent at $P$ to the curve of steepest descent of $|p(z)|$ through $P$ pass through the center of $\Gamma$. Then the circle of curvature of that curve at $P$ cuts $\Gamma$.*

Reciprocally, Theorem 1 is readily proved by use of Theorem 4 (compare [2], Ch. 4).

The invariant form of Theorem 4, readily expressed by the reader, is concerned with the curves of steepest descent of $|r(z)|$, where $r(z)$ is an arbitrary rational function having but one pole in the extended plane.

It might be conjectured that Theorem 1 remains valid without the requirement that the tangent at $P$ to the curve of steepest descent through $P$ should pass through the center of $\Gamma$. This conjecture is false:

THEOREM 5. *There exists an infinite region $R$ with finite boundary $B$ possessing a Green's function $G(x, y)$ with pole at infinity, and a point $P$ in $R$ exterior to a circle $\Gamma$ containing $B$ such that $\Gamma$ is not cut by the circle of curvature at $P$ to the curve of steepest descent of $G(x, y)$ through $P$.*

It will be recalled that [1] the tangent at $P$ to the curve of steepest descent through $P$ must cut $\Gamma$.

By virtue of our discussion of lemniscates, notably method (ii), it is sufficient here to study as defining functions the first members of (13) or (14). We choose $p(z) \equiv z^2 - 1$, and the function conjugate to the harmonic function $\log |p(z)|$ is

$$\arg(z^2 - 1) \equiv \arg(x^2 + 2ixy - y^2 - 1) = \tan^{-1}[(2xy)/(x^2 - y^2 - 1)].$$

The curves of steepest descent of $|p(z)|$ are the curves

$$x^2 - y^2 - 1 + 2cxy = 0, \qquad c = \text{const.}, \tag{15}$$

taken together with the curve $xy=0$, namely the family of equilateral hyperbolas with center $O$: (0, 0) passing through the points $(\pm 1, 0)$. These curves all have the same shape except for the degenerate hyperbola $xy=0$. The asymptotes of the curve (15) are $x^2-y^2+2cxy=0$ with slopes $c\pm(1+c^2)^{1/2}$, and the axes of the curve bisect the angles between the asymptotes, and are therefore given by $cx^2-2xy-cy^2=0$.

As $c\to+\infty$ the asymptotes and the curve (15) itself approach the curve $xy=0$ consisting of the coordinate axes. The slope of (15) in the point (1, 0) is $-1/c$, so for large positive $c$ one branch of the curve (15) passes from the fourth quadrant into the first quadrant at (1, 0) as $x$ decreases, and that branch remains in the first quadrant as $(x, y)$ continues to trace the curve monotonically; the vertex lies near the origin. The circle of curvature of an equilateral hyperbola at a vertex lies in the closed interior of the corresponding branch of the curve; that circle subtends a certain angle $2\gamma$ at the center of the curve which is independent of the size of the curve, $0<\gamma<45°$. Then when $c$ is sufficiently large (and positive) a vertex $P$ of (15) lies in the first quadrant near the origin, and an asymptote (having the slope $c+(1+c^2)^{1/2}$) makes an angle numerically less than $45°-\gamma$ with $Oy$; the circle of curvature at $P$ subtends the angle $2\gamma$ at $O$, so that circle of curvature lies wholly in the first quadrant. Then there exists a new circle $\Gamma$ containing in its interior both the two points $z=\pm 1$ and a suitable lemniscate $B$: $|z^2-1|=A(>0)$, but to which the circle of curvature is exterior. Theorem 5 is established. Likewise in Theorem 2 there cannot be omitted the requirement that the tangent at $P$ to the curve of steepest descent through $P$ shall be perpendicular to $L$.

We have seen in Theorem 5 that Theorem 1 is false if there is omitted the requirement that the tangent at $P$ to the curve of steepest descent should pass through the center of $\Gamma$. Nevertheless, even if that requirement is omitted, and if $\Gamma$ subtends an acute angle at $P$, weaker conclusions can be drawn concerning the circle of curvature, by virtue of the fact that the tangent at $P$ to the curve of steepest descent through $P$ must cut $\Gamma$ (and indeed is not tangent to $\Gamma$). Under these conditions *the circle of curvature must cut at least one of the two lines $L_1$ and $L_2$, each tangent to $\Gamma$, each perpendicular to a tangent from $P$ to $\Gamma$, each separating $P$ from the interior of $\Gamma$*. In fact, let us consider all the lines tangent to $\Gamma$ at points of $\Gamma$ lying on the shorter arc of $\Gamma$ bounded by the points of tangency of $L_1$ and $L_2$. One of these lines (say $L$) is perpendicular to the tangent at $P$ to the curve of steepest descent through $P$, so $L$ satisfies the requirements of Theorem 2; then the circle of curvature cuts $L$ and hence cuts either $L_1$ or $L_2$, since $L_1$ and $L_2$ separate $L$ from $P$ ($L$ cannot coincide with $L_1$ or $L_2$). The conclusion persists if $B$ lies not necessarily in $\Gamma$ but interior to the infinite convex

region bounded by half-lines of the tangents to $\Gamma$ through $P$ and the shorter arc of $\Gamma$ joining their points of tangency.

A more general remark can be made, still with the hypothesis of Theorem 1 except for the requirement that the tangent at $P$ to the curve of steepest descent should pass through the center of $\Gamma$. Let $\{\Gamma_\mu\}$ be a family of circles each containing the interior of $\Gamma$, let $P$ lie exterior to each $\Gamma_\mu$, and suppose each half-line from $P$ cutting $\Gamma$ passes through the center of at least one $\Gamma_\mu$; then the circle of curvature at $P$ of the curve of least descent through $P$ cuts at least one $\Gamma_\mu$.

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2. ———, Interpolation and Approximation by Rational Functions in the Complex Domain, Amer. Math. Soc. Colloq. Publ., vol. 20, 1935.

## DEMOSIAN SYSTEMS OF QUASIGROUPS*

A. SADE, Marseille, France

**1. Introduction. Definitions.** Since the terms which follow have been used with various meanings in the literature, the sense in which they are to be understood in this paper will first of all be made precise.

*Anti-automorphism.* A groupoid $G$ defined on a set $E$ possesses an anti-automorphism $T \in \mathfrak{S}_E$ if the groupoid $G'$, isomorphic to $G$ under the permutation $T$, is conjoint to $G(x \cdot y = z \Leftrightarrow yT \cdot xT = zT)$. ([11], p. 212).

*Conjoint* (*groupoid, expression, equation*). Two groupoids $G = E(\times)$ and $G' = E(*)$, defined on the same set $E$, are conjoint if $\forall\ x, y \in E$, $x \times y = y * x$ ([6], p. 60, no. 75). The symbols $\times$ and $*$ are similarly conjoint. Two expressions are conjoint if one is obtained from the other by replacing the symbols of operation with their conjoint symbols. Two equations are conjoint if one is obtained from the other by replacing the two members of each equation with their conjoint expressions ([8], p. 78, nos. 4.2, 6.1).

*Demosian* (*system, identity*). Given a set $E$ with laws of composition $\phi_i$, $i = 1, 2, \cdots$, forming a set $\Phi$, then $(E, \Phi)$ is said to be a demosian system. If, in this system, a condition such as (17) below is satisfied, then (17) is said to be a demosian identity [10].

*Groupoid.* A groupoid is a set $E$ with a law of composition $\times$ such that to each

* Translated from the French.

pair $\langle x, y\rangle \in E^2$ there corresponds a unique element $z \in E$ called the product of $x$ by $y$ and written $x \times y = z$ ([7], p. 157, no. 3).

*Left inverse.* In the groupoid $G = E(\times)$ with unit $u$, the element $y$ is a left inverse of $x$ if $y \times x = u$.

*Isotopy.* Two groupoids $Q = E(\cdot)$ and $R = E(\times)$, defined on the same set $E$, are isotopic if there exist three permutations of $E$, $(\xi, \eta, \zeta)$ such that $\forall\, x, y \in E$, $(x \cdot y)\zeta = (x\xi) \times (y\eta)$ ([7], p. 164, no. 15).

*Loop.* A loop is a quasigroup with a two-sided unit ([7], p. 164, no. 15).

*Multigroupoid.* A multigroupoid is a set $E$ with an operation $\times$ such that to each pair $\langle x, y\rangle \in E^2$ there corresponds a subset (in general, proper) of $E$, $\varnothing \subseteq x \times y \subseteq E$ ([9], p. 231).

*Quasigroup.* A quasigroup is a groupoid satisfying the axiom that each of the equations $ax = b$ and $ya = b$ has a unique solution for $x$ and $y$, respectively, ([7], p. 15, no. 3; [8], p. 73, no. 1).

*Right translation.* If $Q = E(\times)$ is a groupoid, the right translation by the element $s$ is the mapping $\Delta_s: x \to x \times s$ of $E$ into itself ([1], p. 509).

*Twist.* A twist is an isotopy in which the first two components are the identity, *i.e.*, $T = (1, 1, \zeta)$ ([7], p. 171, no. 28).

*Left unit.* The element $u$ is a left unit of the groupoid $G = E(\cdot)$ if $\forall x \in E$, $u \cdot x = x$ ([13], p. 1).

If the twelve ways of forming the product of three elements $a, b, c$ are equated two at a time, then $\frac{1}{2} \cdot 12 \cdot 11 = 66$ equations are obtained. Of these, 48 have the identity as their group of automorphisms and are isomorphic in sets of 6 to 8 of them. The remaining 18 have a group of automorphisms of the second order and are isomorphic in sets of 3 to 6 of them. The 14 nonisomorphic laws include the usual associative law

$$(0) \qquad (ab)c = a(bc)$$

and the 13 which follow, numbered according to the classification of Faragó ([3], p. 133):

| | | | |
|---|---|---|---|
| (1) | $(ab)c = a(cb)$, | (9) | $a(bc) = c(ab)$, |
| (2) | $(ab)c = b(ac)$, | (10) | $a(bc) = c(ba)$, |
| (3) | $(ab)c = b(ca)$, | (11) | $(ab)c = (ac)b$, |
| (4) | $(ab)c = c(ab)$, | (12) | $(ab)c = (ba)c$, |
| (5) | $(ab)c = c(ba)$, | (14) | $(ab)c = (ca)b$, |
| (6) | $a(bc) = a(cb)$, | (15) | $(ab)c = (cb)a$. |
| (7) | $a(bc) = b(ac)$, | | |

In this list, equations (8) $a(bc) = b(ca)$ and (13) $(ab)c = (bc)a$ have been omitted since they are isomorphic to (9) and (14), respectively, and are deducible from them through the permutation $(abc) \to (bca)$.

Some of these laws are already known under different names ([7], p. 154): (1) = Eingewandte Produkt (22); (7) = left permutability or the law of Hosszú (19); (9) = cyclic associativity (18); (10) = the law of Abel-Grassman (21); (11) = right permutability (20). Faragó studied groupoids with a unit and left inverse satisfying the laws (0)–(15).

Hosszú [4] gave the general solution of these equations for continuous, strictly monotone functions over the real field $\Re$, reducing them to four only, namely (0), (7), (9), and (10). He also solved (p. 212), still over the real field, the equation

$$F[x, G(y, z)] = H[K(x, y), z], \tag{h}$$

where $F$, $G$, $H$, $K$ are continuous and differentiable; equation (h) is more general than (0).

If $\phi_1, \phi_2, \phi_3, \phi_4$ are the symbols of operation of four quasigroups over any set $E$, Belousov [2] announced and Hosszú [5] proved that the demosian equation

$$(x\phi_1 y)\phi_2 z = x\phi_3(y\phi_4 z) \tag{0}$$

has the general solution

$$\begin{aligned} x\phi_1 y &= (x\xi \cdot y\theta)\mu^{-1}, & x\phi_2 y &= (x\mu \cdot y\lambda)\zeta^{-1}, \\ x\phi_3 y &= (x\xi \cdot y\eta)\zeta^{-1}, & x\phi_4 y &= (x\theta \cdot y\lambda)\eta^{-1}, \end{aligned} \tag{I}$$

where $E(\cdot)$ is any group and $\xi, \eta, \zeta, \theta, \lambda, \mu$ are arbitrary permutations of $E$, a result which is easily seen to have a counterpart for multigroupoids [9].

In this paper it is proposed to study the demosian systems which satisfy (0)–(15) and, in addition, each of the equations

$$(x\phi_1 y)\phi_2 z = (x\phi_3 y)\phi_4 z, \tag{16}$$

$$z\phi_1(x\phi_2 y) = z\phi_3(x\phi_4 y); \tag{17}$$

these equations are not of interest as long as the study is concerned with a single groupoid (Faragó), but may no longer be neglected in the demosian case. An approach to the problem is to first of all solve (0)–(17) over a demosian system of quasigroups. It will be seen that these equations reduce to only two.

**2. Solution of (0)–(17).** These equations may be arranged in the following order and in two categories, the first is

| | | | |
|---|---|---|---|
| (7) | $z\phi_1(x\phi_2 y) = x\phi_3(z\phi_4 y)$, | (3) | $(y\phi_1 x)\phi_2 z = x\phi_3(z\phi_4 y)$, |
| (10) | $z\phi_1(y\phi_2 x) = x\phi_3(y\phi_4 z)$, | (2) | $y(\phi_1 x)\phi_2 z = x\phi_3(y\phi_4 z)$, |
| (9) | $z\phi_1(x\phi_2 y) = x\phi_3(y\phi_4 z)$, | (11) | $(y\phi_1 x)\phi_2 z = (y\phi_3 z)\phi_4 x$, |
| (0) | $(x\phi_1 y)\phi_2 z = x\phi_3(y\phi_4 z)$, | (14) | $(y\phi_1 x)\phi_2 z = (z\phi_3 y)\phi_4 x$, |
| (1) | $(x\phi_1 y)\phi_2 z = x\phi_3(z\phi_4 y)$, | (15) | $(x\phi_1 y)\phi_2 z = (z\phi_3 y)\phi_4 x$; |

and the second,

(16) $(x\phi_1 y)\phi_2 z = (x\phi_3 y)\phi_4 z$, (4) $(x\phi_1 y)\phi_2 z = z\phi_3(x\phi_4 y)$,

(12) $(x\phi_1 y)\phi_2 z = (y\phi_3 x)\phi_4 z$, (17) $z\phi_1(x\phi_2 y) = z\phi_3(x\phi_4 y)$,

(5) $(x\phi_1 y)\phi_2 z = z\phi_3(y\phi_4 x)$, (6) $z\phi_1(x\phi_2 y) = z\phi_3(y\phi_4 x)$.

In each of the two series, every equation leads to the one following by replacing an operation $\phi$ by its conjoint. The solution of (0) is already known and, according to the way in which an equation is derived from the preceding one, its solution follows from that of the preceding equation by a technique of which a single example will suffice. For example, let $x\psi y = y\phi_4 x$; then, if $\phi_1, \phi_2, \phi_3, \phi_4$ are solutions of (0), those of (1) will be $\phi_1, \phi_2, \phi_3, \psi$; *i.e.*, permuting $x$ and $y$ in the expression for $\phi_4$,

$$(x\xi \cdot y\theta)\mu^{-1}, \qquad (x\mu \cdot y\lambda)\zeta\lambda^1, \qquad (x\xi \cdot y\eta)\zeta^{-1}, \qquad (y\theta \cdot x\lambda)\eta^{-1}.$$

More generally, if a demosian equation $(D)$ goes into another demosian equation $(D')$ by permuting two factors in $(D)$, then the solution of $(D')$ follows from that of $(D)$ by substituting the conjoint groupoid for the groupoid defined by the operation involved in the composition of the two factors. Table I is obtained in

TABLE I

| | | | | |
|---|---|---|---|---|
| (7) | $(y\mu \cdot x\lambda)\zeta^{-1}$ | $(x\xi \cdot y\theta)\mu^{-1}$ | $(x\xi \cdot y\eta)\zeta^{-1}$ | $(y\theta \cdot x\lambda)\eta^{-1}$ |
| (10) | $(y\mu \cdot x\lambda)\zeta^{-1}$ | $(y\xi \cdot x\theta)\mu^{-1}$ | $(x\xi \cdot y\eta)\zeta^{-1}$ | $(x\theta \cdot y\lambda)\eta^{-1}$ |
| (9) | $(y\mu \cdot x\lambda)\zeta^{-1}$ | $(x\xi \cdot y\theta)\mu^{-1}$ | $(x\xi \cdot y\eta)\zeta^{-1}$ | $(x\theta \cdot y\lambda)\eta^{-1}$ |
| (0) | $(x\xi \cdot y\theta)\mu^{-1}$ | $(x\mu \cdot y\lambda)\zeta^{-1}$ | $(x\xi \cdot y\eta)\zeta^{-1}$ | $(x\theta \cdot y\lambda)\eta^{-1}$ |
| (1) | $(x\xi \cdot y\theta)\mu^{-1}$ | $(x\mu \cdot y\lambda)\zeta^{-1}$ | $(x\xi \cdot y\eta)\zeta^{-1}$ | $(y\theta \cdot x\lambda)\eta^{-1}$ |
| (3) | $(y\xi \cdot x\theta)\mu^{-1}$ | $(x\mu \cdot y\lambda)\zeta^{-1}$ | $(x\xi \cdot y\eta)\zeta^{-1}$ | $(y\theta \cdot x\lambda)\eta^{-1}$ |
| (2) | $(y\xi \cdot x\theta)\mu^{-1}$ | $(x\mu \cdot y\lambda)\zeta^{-1}$ | $(x\xi \cdot y\eta)\zeta^{-1}$ | $(x\theta \cdot y\lambda)\eta^{-1}$ |
| (11) | $(y\xi \cdot x\theta)\mu^{-1}$ | $(x\mu \cdot y\lambda)\zeta^{-1}$ | $(x\theta \cdot y\lambda)\eta^{-1}$ | $(y\xi \cdot x\eta)\zeta^{-1}$ |
| (14) | $(y\xi \cdot x\theta)\mu^{-1}$ | $(x\mu \cdot y\lambda)\zeta^{-1}$ | $(y\theta \cdot x\lambda)\eta^{-1}$ | $(y\xi \cdot x\eta)\zeta^{-1}$ |
| (15) | $(x\xi \cdot y\theta)\mu^{-1}$ | $(x\mu \cdot y\lambda)\zeta^{-1}$ | $(y\theta \cdot x\lambda)\eta^{-1}$ | $(y\xi \cdot x\eta)\zeta^{-1}$ |

this way. In order to solve (16), suppose that $z$ is a constant $a$. Then, if $\Delta_a^4$ denotes the right translation $(x \to x\phi_4 a)$ by $a$ in the quasigroup $E(\phi_4)$,

$$(x\phi_1 y)\Delta_a^2 = (x\phi_3 y)\Delta_a^4, \qquad x\phi_3 y = (x\phi_1 y)\Delta_a^2(\Delta_a^4)^{-1}.$$

Thus if $x\phi_1 y = x \times y$ is any quasigroup, $x\phi_3 y$ is isotopic to $x \times y$ under the twist $(1, 1, \eta)$, *i.e.*, $x\phi_3 y = (x \times y)\eta^{-1}$.

Now let $x\phi_3 y = t$; then $x \times y = t\eta$ and, if $x\phi_2 y = x * y$ is again an arbitrary quasigroup, $\forall t \in E$, $t\eta\phi_2 z = t\phi_4 z$, $x\phi_4 y = x\eta * y$.

The general solution of (16) does not enter into that of (h) and depends on two arbitrary quasigroups and an arbitrary permutation $\eta \in \mathfrak{S}_E$. The same method as that used for Table I yields Table II.

TABLE II

| | | | | |
|---|---|---|---|---|
| (16) | $x \times y$ | $x * y$ | $(x \times y)\eta^{-1}$ | $x\eta * y$ |
| (12) | $x \times y$ | $x * y$ | $(y \times x)\eta^{-1}$ | $x\eta * y$ |
| (5) | $x \times y$ | $x * y$ | $y\eta * x$ | $(y \times x)\eta^{-1}$ |
| (4) | $x \times y$ | $x * y$ | $y\eta * x$ | $(x \times y)\eta^{-1}$ |
| (17) | $y * x$ | $x \times y$ | $y\eta * x$ | $(x \times y)\eta^{-1}$ |
| (6) | $y * x$ | $x \times y$ | $y\eta * x$ | $(y \times x)\eta^{-1}$ |

*Remarks.* (i) *If the solutions of a demosian equation are quasigroups isotopic to the same arbitrary group G, then another solution may be deduced from a given solution by replacing G by a loop isotopic to the conjoint of G.* (It is known ([1], p. 511, Th. 2) that every loop isotopic to a group is again a group.)

(ii) The conjoint of an expression (equation) has been defined as the expression (equation) obtained by replacing the symbols of operation with their conjoint symbols. Thus the equation conjoint to (0) is $(z\psi_4 y)\psi_3 x = z\psi_2(y\psi_4 x)$, where $\psi_1, \psi_2, \psi_3, \psi_4$, are conjoint, respectively, to $\phi_1, \phi_2, \phi_3, \phi_4$. It is clear that, *if two equations are conjoint, then the solutions of the second are conjoint to those of the first.* It is easily verified that (0), (3), (4), (5) are auto-conjoint, that (1) is conjoint to (2), (6) to (12), (7) to (11), (9) to (14), (10) to (15), and (16) to (17). For example, the solutions of (7) are

$$(y\mu \cdot x\lambda)\zeta^{-1}, \qquad (x\xi \cdot y\theta)\mu^{-1}, \qquad (x\xi \cdot y\eta)\zeta^{-1}, \qquad (y\theta \cdot x\lambda)\eta^{-1}.$$

Their conjoint solutions are

$$(x\mu \cdot y\lambda)\zeta^{-1}, \qquad (y\xi \cdot x\theta)\mu^{-1}, \qquad (y\xi \cdot x\eta)\zeta^{-1}, \qquad (x\theta \cdot y\lambda)\eta^{-1},$$

and, putting them in order, they are precisely those of (11). In the same way, taking the conjoint solutions of (1) leads to

$$(y\xi \cdot x\theta)\mu^{-1}, \qquad (y\mu \cdot x\lambda)\zeta^{-1}, \qquad (y\xi \cdot x\eta)\zeta^{-1}, \qquad (x\theta \cdot y\lambda)\eta^{-1}.$$

These four quasigroups, taken in reverse order do indeed satisfy (2). In addition, it may be shown that these solutions are isomorphic to those in the second line of Table I. Let $G(\times)$ be the conjoint of the group $(\cdot)$ so that $x \times y = y \cdot x$. The above solutions become

$$(y\lambda \times x\theta)\eta^{-1}, \quad (x\eta \times y\xi)\zeta^{-1}, \quad (x\lambda \times y\mu)\zeta^{-1}, \quad (x\theta \times y\xi)\mu^{-1}.$$

The permutation $(\lambda\mu\eta\xi)\rightarrow(\xi\eta\mu\lambda)$ of the components of the isotopy leads to

$$(y\xi \times x\theta)\mu^{-1}, \quad (x\mu \times y\lambda)\zeta^{-1}, \quad (x\xi \times y\eta)\zeta^{-1}, \quad (x\theta \times y\lambda)\eta^{-1},$$

which is the solution of (2).

It is clear that the way in which the four solutions are rearranged after passing to the conjoint equations depends on the fact that no distinction is made between the equation $A=B$ and the equation $B=A$.

**3. Demosian systems.** Every quasigroup which is isotopic to a group is isotopic to the conjoint of this group since every group is isomorphic to its conjoint ([6], p. 60, no. 75). If $T$ is the anti-automorphism which transforms $G(\cdot)$ into its conjoint, then, for example, $(y\xi\cdot x\theta)\mu^{-1}=(x\theta T\cdot y\xi T)T^{-1}\mu^{-1}$. Consequently, all the formulas in Table I may be written as isotopes of the same group $G$.

Consider a demosian system of quasigroups defined on a set $E$ by the laws of composition $\phi_i$, $i=1, 2, \cdots$, forming a set $\Phi$, and satisfying any one of the laws (L) of the first category (0)–(15). Let $\phi_1$ be fixed and let $\phi_2$ range over $\Phi$. For each choice of $\phi_2$ there exist $\phi_3, \phi_4\in\Phi$ satisfying (L) and each time the four quasigroups are isotopic to the same group. Since the isotopy is transitive, all the groups under consideration, being isotopic to $E(\phi_1)$, are isotopic (and hence isomorphic) to each other ([1], p. 511). On the other hand, as $\phi_2$ ranges over $\Phi$, $E(\phi_2)$ is always isotopic to $E(\phi_1)$ and hence to a fixed group $G=E(\cdot)$. This gives the following necessary condition:

*Every system of quasigroups $(E, \Phi)$ satisfying any one of the demosian laws of the first category consists of quasigroups isotopic to the same group $G=E(\cdot)$.*

*Example.* The set of quasigroups defined over any field by the law of composition $x\times y=ax+by+c$; $a, b\neq 0$, satisfies every demosian law of the two categories. The demosian subsets with the same property are obtained by taking $a=1$, or $b=1$, or $c=0$, or two of these conditions simultaneously. Each quasigroup $Q=E(\times)$ is isotopic to the additive group $f(x, y)=x+y$ of the field under the isotopy with components $\xi: x\rightarrow ax$, $\eta: x\rightarrow bx$, $\zeta: x\rightarrow x-c$. Each of the seven demosian sets and subsets described above has the property of being a group with respect to the composition of isotopies. This fact suggests the following two questions:

QUESTION I. *What are the demosian laws which define a set of quasigroups isotopic to the same group?*

QUESTION II. *Among these laws what are the ones for which the set of isotopies which define the quasigroups of a demosian set $(E, \Phi)$ is a group with respect to the composition of isotopies?*

It is not easy to state a sufficient condition to impose on the isotopes of the

same group in order that they form a demosian domain; this involves the group of autotopies of a group.

**4. Systems of the second category.** Since the quasigroups $E(\phi_1)$ and $E(\phi_2)$ may be chosen arbitrarily, there is no one condition which can be imposed on two elements of a demosian system in order that they should appear in one of the laws in Table II. For example, such a system may be generated by two quasigroups $E(\times)$, $E(*)$, and a permutation $\eta$ of $E$. Thus, in the case of (16), the general term of $S(E, \Phi)$ would be $(x\eta^m \times y)\eta^{-p}$, $(x\eta^q \times y)\eta^{-r}$. If $\eta$ is finite of order $n$, the system contains a finite number, $2n^2$ of quasigroups.

*Special case.* Putting $\phi_1=\phi_3$, $\phi_2=\phi_4$ in (0) yields the identity,

$$(0_1) \qquad \forall\, x, y, z \in E, \qquad (x\phi_1 y)\phi_2 z = x\phi_1(y\phi_2 z),$$

between $E(\phi_1)$ and $E(\phi_2)$, which has been considered in [11]. The groupoids $E(\phi_1)$ and $E(\phi_2)$, no longer necessarily quasigroups, which satisfy $(0_1)$ are said to be in an associative relation (*en relation associative*, written $\phi_1(\mathbf{RA})\phi_2$). It is easy to see that *this relation is transitive*. For, if $\phi_1(\mathbf{RA})\phi_2$ and $\phi_2(\mathbf{RA})\phi_3$, then, since every element in a groupoid $E(\phi_2)$ may be represented as the product of two elements, say, $u=y\phi_2 z$, it follows that $\forall x, u, t \in E$,

$$\begin{aligned}(x\phi_1 u)\phi_3 t &= [x\phi_1(y\phi_2 z)]\phi_3 t = [(x\phi_1 y)\phi_2 z]\phi_3 t \\ &= (x\phi_1 y)\phi_2(z\phi_3 t) = x\phi_1[y\phi_2(z\phi_3 t)] \\ &= x\phi_1[(y\phi_2 z)\phi_3 t)] = x\phi_1(u\phi_3 t).\end{aligned}$$

Hence $\phi_1(\mathbf{RA})\phi_3$.

Suppose from now on that the $\phi$ are the laws of quasigroups. If $E(\phi_1)$ and $E(\phi_2)$ satisfy $(0_1)$, the four quasigroups $E(\phi_1)$, $E(\phi_2)$, $E(\phi_1)$, $E(\phi_2)$ are isotopes of the same group $G(\cdot)$ according to the formulas (I). Then, denoting the isotopy with components $X$, $Y$, $Z$ by $(X, Y, Z)$,

$$G(\xi, \theta, \mu) = G(\xi, \eta, \zeta), \qquad G(\mu, \lambda, \zeta) = G(\theta, \lambda, \eta),$$

from which it follows that $G(1, \theta\eta^{-1}, \mu\zeta^{-1}) = G = G(\mu\theta^{-1}, 1, \zeta\eta^{-1})$. Moreover, $(1, \theta\eta^{-1}, \mu\zeta^{-1})$ and $(\mu\theta^{-1}, 1, \zeta\eta^{-1})$ are two autotopies of $G$. The solutions $\phi_1$, $\phi_2$ are found by choosing two autotopies of $G(\cdot)$ of the form $(1, Y, Z)$ and $(X', 1, Z')$. The components $\xi, \theta, \mu, \cdots$, of (I) will then be determined by the equations

$$\theta\eta^{-1} = Y, \qquad \mu\zeta^{-1} = Z, \qquad \mu\theta^{-1} = X', \qquad \zeta\eta^{-1} = Z',$$

from which it follows, taking $\xi$, $\eta$, $\lambda$ arbitrarily, that $\theta = Y\eta$, $\zeta = Z'\eta$, $\mu = ZZ'\eta$, with the condition $\mu\theta^{-1} = ZZ'\eta\eta^{-1}Y^{-1} = X'$ or $ZZ' = X'Y$. Hence:

*If two quasigroups $E(\phi_1)$ and $E(\phi_2)$ satisfy the demosian equation $(0_1)$, they are isotopic to the same group $G$ under $(\xi, \eta, Z'\eta)$ and $(Y\eta, \lambda, \eta)$, respectively, where $\xi, \eta, \lambda$ are any permutations of $E$, and $(1, Y, Z)$, $(X', 1, Z')$ are two autotopies of $G$ satisfying the condition $ZZ' = X'Y$.*

A demosian domain $(E, \Phi)$ of quasigroups satisfying the identity,

$$(18) \qquad \forall x, y, z \in E, \quad \forall \phi_1 \in \Phi, \quad \exists \phi_2 \in \Phi, \quad \phi_1(\mathbf{RA})\phi_2,$$

may be constructed in the following way. Let $Y$ and $Z'$ be fixed. Then, starting with two quasigroups derived from $G(\cdot)$ under the isotopies $(\xi, \eta, Z'\eta)$ and $(Y\eta, \lambda, \eta)$, a third quasigroup $E(\phi_3)$ is constructed such that $\phi_2(\mathbf{RA})\phi_3$ by taking $Y\eta = \xi_1$, $\lambda = \eta_1$, $\eta = Z'\eta_1$. Then $E(\phi_3)$ will be isotopic to $G$ under $(Y\eta_1, \lambda_1, \eta_1) = (Y\lambda, \lambda_1, Z'^{-1}\eta)$ with the condition $Z'^{-1}\eta = \eta_1 = \lambda$. This requires that the second isotopy $\phi_2$ be chosen in the form $(Y\eta, Z'^{-1}\eta, \eta)$; the third isotopy is then $(YZ'^{-1}\eta, Z'^{-2}\eta, Z'^{-1}\eta)$ since $\lambda_1 = Z'^{-1}\eta_1$; the fourth is $(YZ'^{-2}\eta, Z'^{-3}\eta, Z'^{-2}\eta)$, etc. If $Z'$ is finite of order $k$, the demosian domain will contain $k$ other quasigroups besides $E(\phi_1)$. The first, $E(\phi_1)$ will be (**RA**) to all the others; the $k$ remaining ones forming a circular chain, will each be (**RA**) to all the others and, for these $k$ quasigroups, the associative relation (**RA**) will then be reflexive and symmetric. Furthermore, each of them will be mutually associative to the $k-1$ other quasigroups and will itself be a semigroup [11] and hence a group isomorphic to $G$. The argument is no longer valid if $Z'$ is not of finite order since then the associative relation (**RA**) is only transitive. In this case a direct proof is necessary. It is known ([1], p. 510, Th. I) that the necessary and sufficient condition for the isotope $G'(*)$ of a loop $G(\cdot)$ to be again a loop under $(\xi, \eta, \zeta)$, $x\xi * y\eta = (x \cdot y)\zeta$, is that the permutations $\xi\zeta^{-1}$ and $\eta\zeta^{-1}$ be respectively a right and a left translation of $G(\cdot)$.

Since $(1, Y, Z)$ is an autotopy of $G(\cdot)$, the permutations $Z^{-1}$ and $YZ^{-1}$ are the translations $Z^{-1} = \Delta_b$, $YZ^{-1} = \Gamma_a$, so that $Z = \Delta_b^{-1}$, $Y = \Gamma_a\Delta_b^{-1}$. Similarly, since $(X', 1, Z')$ is also an autotopy, $Z' = \Gamma_c^{-1}$, $X' = \Delta_d\Gamma_c^{-1}$. The condition $ZZ' = X'Y$ may then be expressed in the form

$$\Delta_b^{-1}\Gamma_c^{-1} = \Delta_d\Gamma_c^{-1}\Gamma_a\Delta_b^{-1}, \qquad \Delta_b = \Gamma_c\Delta_b\Delta_d\Gamma_c^{-1}\Gamma_a.$$

Then $\forall x \in G$,

$$x \cdot b = (c \cdot x \cdot b \cdot d)\Gamma_c^{-1}\Gamma_a = a \cdot x \cdot b \cdot d, \quad x \cdot b \cdot d^{-1} \cdot b^{-1} = a \cdot x = x\Gamma_a.$$

Finally, $xY = x\Gamma_a\Delta_b^{-1}$ or $xY\Delta_b = x\Gamma_a = x \cdot b \cdot d^{-1} \cdot b^{-1}$, $xY = x \cdot b \cdot d^{-1} \cdot b^{-2}$, $Y = \Delta_{b \cdot d^{-1} \cdot b^{-2}}$. Thus $Y$ is a right translation of $G(\cdot)$. Similarly, $Z'^{-1} = \Gamma_c$ is a left translation. By the lemma cited above, the isotope of $G(\cdot)$ under $(Y, Z'^{-1}, 1)$ is a loop. But a loop which is isotopic to a group is necessarily a group isomorphic to that group 1. It has therefore been shown that, *if* $(1, Y, Z)$ *and* $(X', 1, Z')$ *are two autotopies of a group* $G$ *satisfying the condition* $ZZ' = X'Y$, *then the isotope of* $G$ *under* $(Y, Z'^{-1}, 1)$ *is a group isomorphic to* $G$.

If this is applied to the demosian system constructed as above by starting with a group $G(\cdot)$ and two isotopes $E(\phi_1)$ and $E(\phi_2)$, it is seen that *the system* $(E, \Phi)$, *satisfying* (18) *and generated by two isotopes* $E(\phi_1)$, $E(\phi_2)$ *of* $G$, *is always composed, in addition to* $E(\phi_1)$, *of groups isomorphic to* $G$, *each of which is* (**RA**) *to all the rest.*

*Example* I. Let $G=\mathfrak{S}_3=012345$, $G\cdot 1=120534$, $G\cdot 2=201453$, $G\cdot 3=345012$; $Y=(102)(345)=Z$, $X'=Z'=(102)(354)$. Taking $\eta=1$ it is found that $\phi_1$: $(\xi, 1, Z')$, $\phi_2$: $(Y, Z'^{-1}, 1)$, $\phi_3$: $(YZ'^{-1}, Z'^{-2}, Z'^{-1})$, $\phi_4$: $(YZ'^{-2}, 1, Z'^{-2})$. The latter three are isomorphic to $\mathfrak{S}_3$ under $YZ'$, $Y$, and $YZ'^{-1}$, respectively.

*Example* II. Consider the additive group $G(\cdot)$, $x\cdot y=x+y$ of the ring $\mathfrak{Z}/n$. It is found that every pair of autotopies of the cyclic group satisfying $ZZ'=X'Y$ is of the form $X=1$, $Y=Z=(x\to x+a)=A$, $X'=(x\to x+b)=Z'=B$, $Y'=1$. Then $\phi_1$: $(\xi, \eta, B^{-1}\eta)$,

$$\phi_{i+1}\text{: } (AB^{i-1}\eta,\ B^i\eta,\ B^{i-1}\eta) \qquad (i = 1, 2, \cdots),$$

where $\phi_i$ is isomorphic to $G$ under the permutation $AB^{i-1}\eta$. The demosian domain contains $E(\phi_1)$ and $k$ groups isomorphic to $G$, where $k$ is the order of $B$.

If it is no longer assumed that the demosian system is made up only of quasigroups, such a domain may be constructed by application of the general methods for the formation of groupoids (**RA**) [11].

A study analogous to this one may be made for each of the 16 equations of Section 2.

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## THE ROLLING OF ONE CURVE OR SURFACE UPON ANOTHER

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It is well known in dynamics [1] that any general motion of a lamina in its own plane may be generated by the rolling of one curve upon another. In his book on continuous groups, Kowalewski [2] gives a mathematical treatment of this result, and here, by using matrix methods, we shall generalize his theorem to spaces of higher dimensions.

The motion of a body in Euclidean space of $n$ dimensions is generated by the transformation

$$X = AX_0 + C, \tag{1}$$

where $A$ is a proper orthogonal $n \times n$ matrix, and $X_0$ and $C$ are $n \times 1$ vectors. The elements of $A$ and $C$ are continuously differentiable functions of $t$, and the elements of $X_0$ are the coordinates of a point in the body at the initial time, $t = t_0$. To avoid the cases of pure translation and pure rotation we assume that $\dot{A} \neq 0$, $\dot{C} \neq 0$, where $\dot{A} = dA/dt$. We shall discuss whether such motions may be considered as generated by the rolling of one curve or surface upon another curve or surface fixed in space.

In a motion generated by the rolling of one curve upon another, the point of contact is an instantaneous center of rotation for the rolling curve. For those points which are instantaneous centers of rotation at time $t$, we must have $\dot{X} = 0$. Hence, in order to find the position at time $t_0$ of those points which will become instantaneous centers at time $t$, we must solve the equations

$$0 = X = AX_0 + C \tag{2}$$

for the vector $X_0$. The equations

$$AX_0 = -\dot{C} \tag{2a}$$

are uniquely solved by

$$X_0 = -(A)^{-1}C \tag{3}$$

if $\dot{A}$ is nonsingular. However, since $A^TA = I$, we have

$$0 = A^TA + A^TA = (A^TA)^T + (A^TA),$$

and so $(A^T\dot{A})$ is an antisymmetric $n \times n$ matrix with zero determinant if $n$ is odd, and except when its pfaffian vanishes, with nonzero determinant when $n$ is even. A motion in Euclidean space of even dimensions for which the pfaffian of $(A^T\dot{A})$ vanishes shall be called a *singular* motion. In $E_n$, if $n$ is even, and if the motion is not singular, the initial positions of those points which at time $t$ become instantaneous centers of rotation, are given by (3). At time $t$, these are transformed into the points

$$X = -A(A)^{-1}C + C \tag{4}$$

by (1). Thus the *rolling curve* is given by (3), and the *curve rolled upon* by (4), when the motion (1) is not singular and takes place in Euclidean space of even dimensions.

The cases in which $n$ is odd, or in which $n$ is even and the motion is singular, need further consideration. Here, $\det(A^T\dot{A}) = \det(\dot{A}) = 0$, and so equations (2a) are solvable if and only if

$$(A^T)Y = 0 \Rightarrow (C^T)Y = 0. \tag{5}$$

In general, these conditions will not be satisfied for an arbitrary motion. We shall now develop those physical conditions necessary to insure solvability of (2a), or equivalently, to interpret the motion as the rolling of one surface upon another.

We consider an infinitesimal motion of a body, given by (1), with the consistency conditions (5) imposed. For an infinitesimal displacement of the body,

$$X + \dot{X}\delta t = AX_0 + C + (AX_0 + \dot{C})\delta t. \tag{6}$$

Since $\det(A) \neq 0$, $A^{-1}$ exists, and we may solve (1) for $X_0$: $X_0 = A^{-1}(X - C)$. Substituting this expression in (6), we obtain

$$X + X\delta t = C + [(I + \dot{A}A^T\delta t)(X - C)] + \dot{C}\delta t. \tag{7}$$

Since the matrix $(I + \dot{A}A^T\delta t)$ is the matrix of an infinitesimal orthogonal transformation [3], equations (7) represent an infinitesimal rotation on the vector $(X - C)$, followed by an infinitesimal translation, $\dot{C}\delta t$.

Now, in any rotation, the points on the axis of rotation remain fixed. Hence, if $Y$ is a vector along the axis of the rotation represented by the matrix $(I + \dot{A}A^T\delta t)$ then

$$(I + \dot{A}A^T\delta t)Y = Y,$$

which implies that $\dot{A}A^TY = 0$. Since $AA^T = I$ yields $\dot{A}A^T = -A\dot{A}^T$, we have $A\dot{A}^TY = 0$ or $\dot{A}^TY = 0$.

Thus we see that the condition (5) states that the infinitesimal translation $\dot{C}\delta t$ must be perpendicular to any vector $Y$ which lies along the axis of infinitesimal rotation. *Therefore, for any motion in Euclidean space of odd dimensions, and for singular motions in space of even dimensions, a necessary and sufficient condition for finding $X_0$, or the center of instantaneous rotation, is that the infinitesimal translation at any moment be perpendicular to the axis of infinitesimal rotation.*

If we suppose that condition (5) is fulfilled and that the matrix $A$ in (2) has rank $r$, then the solution of (2) is

$$X_0 = Q_0 + \sum_{i=1}^{r} \lambda_i P_i, \tag{8}$$

where $r+s=n$. Here $Q_0$ is a particular solution of (2); the $P_i$ are the $s$ linearly independent solutions of the homogeneous equation, and the $\lambda_i$ are parameters. Hence we obtain, not a curve, but an $(s+1)$-dimensional manifold generated by $s$-dimensional flats, as the rolling manifold. The manifold rolled upon is

$$X = AX_0 + C, \tag{9}$$

where $X_0$ is given by (8). Thus the situation is similar to that in three-dimensional space.

It seemed interesting to investigate if any particular curve upon the manifold in (9) deserved to be considered as that curve upon which the rolling takes place, so that the picture would again be that of one curve rolling upon another as in a space of even dimensions. As the motion is not one of pure rotation, there is no fixed center of rotation. However, we can demand that the movement of the center of rotation should be minimized, and this is achieved if we choose on the manifold that curve whose velocity at each instant has minimum absolute value. A curve on the manifold is obtained by making the $\lambda_i$ functions of $t$ in (8) and (9), and we now wish to minimize

$$|\dot{X}|^2 = |AX_0 + \dot{C} + A\dot{X}_0|^2 = |A\dot{X}_0|^2 = |\dot{X}_0|^2$$

for variable $\lambda_i$ and $\dot{\lambda}_i$. Thus we find the equations

$$P_i^T\left\{\dot{Q}_0 + \sum_{j=1}^{s}(\dot{\lambda}_j P_j + \lambda_j P_j)\right\} = 0, \qquad (i = 1, \cdots, s) \tag{10a}$$

$$P_i^T\left\{Q_0 + \sum_{j=1}^{s}(\dot{\lambda}_j P_j + \lambda_j \dot{P}_j)\right\} = 0, \tag{10b}$$

and these yield values for the $\lambda_i$ and $\dot{\lambda}_i$. It is not true, in general, that when we solve (10a) and (10b) for $\lambda_i$ and $\dot{\lambda}_i$ the value obtained for $\dot{\lambda}_i$ is the derivative of the value obtained for $\lambda_i$. However, the values found for the $\lambda_i$ are those which determine the *line of striction* on the manifold, and it has long been recognized [4] that the line of striction is not in general perpendicular to the generating flats, or equivalently, that the $\lambda_i$ and their derivatives need not satisfy (10a). By using these values for the $\lambda_i$ in (8) and (9), the lines of striction are obtained as corresponding curves on the two manifolds, and we consider these two curves to have intrinsic interest for the motion.

We shall illustrate the above theory with two simple examples. Consider the motion in the Euclidean plane generated by the transformation of the type (1), when

$$A = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}, \qquad C = \begin{bmatrix} t \\ 0 \end{bmatrix},$$

where in the matrix $A$, $c$ and $s$ denote $\cos t$ and $\sin t$. Here $\det(A) = 1$, and so the rolling curve $X_0$ is, by (3),

$$X_0 = \begin{bmatrix} s & c \\ -c & s \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ -c \end{bmatrix},$$

which is a circle. The curve rolled upon is, by (4),

$$X = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} s \\ -c \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ -1 \end{bmatrix},$$

which is a straight line.

For the second example we consider a motion in Euclidean space of three dinensions generated by

$$A = (1/a)\begin{bmatrix} ac & 0 & as \\ -s^2 & -1 & sc \\ -s & s & c \end{bmatrix}, \qquad C = (1/a)\begin{bmatrix} at \\ s \\ 1-a \end{bmatrix},$$

where $a=\sqrt{(1+s^2)}$. In this case

$$A = -(1/a^3)\begin{bmatrix} a^3 s & 0 & -a^3 c \\ 2sc+s^3 c & -sc & -c^2+s^2+s^4 \\ c & -c & s+s^3+sc^2 \end{bmatrix}, \qquad \dot{C} = (1/a^3)\begin{bmatrix} a^3 \\ c \\ -sc \end{bmatrix},$$

and $\det(\dot{A})=0$. However, this is not a general motion as the conditions (5) are satisfied. The general solution to (2) is, as in (8),

$$X_0^T = [s+\lambda c^2, \lambda(1+s^2), -c+\lambda sc],$$

which is the rolling surface, and the surface rolled upon is, by (9)

$$X^T = [t+\lambda c, -\lambda a, -1+\lambda as].$$

In both of the cases we obtain the line of striction by giving $\lambda$ the value $(3s+2s^3-s^5)/(2+8s^2-6s^4)$ in accordance with (10a) and (10b).

The theory presented in this paper may be slightly generalized by taking as the fundamental form preserved under the transformations, not $\sum_{i=1}^n x_i^2$, but a nondegenerate quadratic form, $\sum_{i=1}^n \sum_{j=1}^n c_{ij}x_i x_j$.

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# REMARKS ON A MULTIVARIATE GAMMA DISTRIBUTION

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**1. Summary.** In this note, the uniqueness problem of a "natural" formulation of a multivariate gamma distribution (Th. 2) is considered. Also the usefulness of the knowledge of the arithmetical character of distributions (in the sense of P. Lévy [6]), concerning extensions of domains of certain parameters (Sec. 4) is discussed.

**2. A multivariate gamma distribution.** Let $\mathbf{Z}=(Z_1, \cdots, Z_p)$ be a vector of $p$ correlated random variables, the density of the marginal distribution function (d.f.) of each of which is

$$(1) \qquad f(x) = \frac{x^{\beta-1}e^{-x/\alpha}}{\alpha^{\beta}\Gamma(\beta)} \qquad \text{if } x > 0, \text{ and zero otherwise,}$$

where $\alpha, \beta > 0$ are real numbers. Now let us consider a vector $\mathbf{X}_u=(X_{1u}, \cdots, X_{pu})$ distributed as a $p$-variate normal vector $(u=1, \cdots, n)$ with mean vector zero and covariance matrix $M$. Writing for each $i$, $Z_i=\sum_{u=1}^{n} X_{iu}^2$, where $M=(\sigma_{ij})$, we see that $Z_i$ is distributed as (1) with $\alpha_i=2\sigma_{ii}$ and $\beta=\frac{1}{2}n$. Then a "natural" formulation of the distribution of $\mathbf{Z}$ is this. Since for all "$i$" the $X_i$'s (hence $Z_i$'s), $(i=1, \cdots, p)$, are correlated, a possible distribution of $(Z_1, \cdots, Z_p)$ is obtained from a multivariate normal d.f. ([5], [3]). It is no restriction to take $M$ as $\sigma^2 P$, where $P$ is the correlation matrix and $\sigma^2>0$ is a real constant, and one obtains the characteristic function (ch.f.) of the d.f. of $\mathbf{Z}$ from that of $\mathbf{X}$ as (cf. [1] p. 159, for the derivation of the ch.f. of the Wishart distribution and then see [2], p. 297),

$$(2) \quad \phi(t_1, \cdots, t_p) = E[\exp\{i(t_1Z_1 + \cdots + t_pZ_p)\}] = |I - 2i\sigma^2 D_t P|^{-n/2},$$

where $D_t$ is a diagonal matrix whose elements are $(t_1, \cdots, t_p)$.

We observe immediately that (2) becomes the ch.f. of the Wishart distribution if $D_t$ is replaced by $D$, where $D$ is a full (symmetric) matrix of $t$'s. Hence setting all the off-diagonal elements of $D$ in the ch.f. of the latter d.f. equal to zero, we get (2), *i.e.*, (2) is precisely the ch.f. of a marginal distribution of the Wishart distribution.

Thus, we may state this observation in terms of the following

THEOREM 1. *Let $\alpha=2\sigma^2$, and $\beta>0$ be real members. For $\beta=\frac{1}{2}n$, $n$ an integer, the multivariate gamma distribution is a marginal distribution of the diagonal elements of a Wishart matrix and its ch.f. is given by* (2).

*Note.* This statement does not assert that (2) holds for any real $\beta>0$, nor does it imply that the choice of the Wishart distribution is determined by $\alpha, \beta>0$, alone. These points are examined in some detail in the following sections.

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**3. Relations between the parameters of the gamma and the "accompanying" Gaussian distributions.** Consider the relations between the correlations in the two distributions. For simplicity we assume in the following, with the $p$-variate normal, the $p$-variate gamma distribution is nonsingular.

LEMMA 1. *Let $K=(k_{ij})$ be the correlation matrix of a gamma distribution and $P=(\rho_{ij})$ be that of the "accompanying" normal distribution. Then for all $\alpha, \beta>0$, consistent with the conditions on these distributions, the relation*

$$k_{ij} = \rho_{ij}^2 \qquad (i, j = 1, \cdots, p)$$

*holds.* ($\rho_{ii}=1=k_{ii}$).

Thus the correlations between any two variates of a gamma distribution defined by (2) are nonnegative. (*cf.* [2], p. 317.)

*Proof.* Writing out (2) in which we set $\beta=\frac{1}{2}n$ and $\alpha=2\sigma^2$, we have the determinant

$$\Delta = \begin{vmatrix} 1-i\alpha t_1\rho_{11} & -i\alpha t_1\rho_{12}\cdots & -i\alpha t_1\rho_{1p} \\ -i\alpha t_2\rho_{12} & 1-i\alpha t_2\rho_{22}\cdots & -i\alpha t_2\rho_{2p} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -i\alpha t_p\rho_{1p} & -i\alpha t_p\rho_{2p}\cdots & 1-i\alpha t_p\rho_{pp} \end{vmatrix}$$

$=1-y$, say. Taking $t_1, \cdots, t_p$ sufficiently small, we can make $|y|<1$ so that

$$\phi(t_1, \cdots, t_p) = \Delta^{-\beta} = 1 + i\alpha\beta \sum t_i\rho_{ii} + \tfrac{1}{2}i^2\alpha^2\beta(\beta+1)\sum t_i^2\rho_{ii}^2 + i^2\beta\alpha^2 \sum\sum t_it_j[(\rho_{ij}^2 - \rho_{ii}\rho_{jj}) + (\beta+1)\rho_{ii}\rho_{jj}] + \cdots, \tag{3}$$

where the summation ranges are $i, j=1, \cdots, p$, $i\neq j$.

From (3) by a routine computation which involves taking partial derivatives with respect to $t_i$, $t_j$, and setting them equal to zero in the resulting expressions, it follows that $k_{ij}=\rho_{ij}^2$ $(i, j=1, \cdots, p)$ as stated in the lemma.

*Remark.* Since $\alpha, \beta$ do not appear in the final result (even in the proof we did not use the fact that $\beta=\frac{1}{2}n$), the result is valid for all real $\alpha, \beta>0$.

One may, at this point, wish to find the corresponding relations between the correlations of the Wishart and normal distributions. All the correlations associated with the "square terms" will be the same as above, but the remaining $\frac{1}{8}p(p-1)(p^2+3p-2)$ correlations have complicated expressions and hence are omitted.

THEOREM 2. *Let $K=(k_{ij})$ be the covariance matrix of a $p$-variate gamma distribution and $\alpha, \beta$ be the other two parameters as before. If $h_i$ $(i=1, \cdots, p-1)$ are the number of nonzero elements (i.e., correlations) in the $i$th row above the main diagonal of $K$ and $q=h_1+\cdots+h_{p-1}$, then, for given $\alpha, \beta>0$, and $K$ (symmetric*

*positive definite* (p.d.)) *with* $k_{ij}$ *small* ($<(p-1)^{-2}$, *say*), *there exist* $2^q$ *distinct p-variate gamma distributions, all of whose ch.f.'s are of the form* (2), *and have the same* $\alpha$, $\beta$ *and* $K$.

*Proof.* Let $P=(\rho_{ij})$, where $\rho_{ij}=\pm\sqrt{k_{ij}}$ $(i, j=1, \cdots, p)$. Of course $k_{ij}$ are all positive (or zero) so that $\rho_{ij}$ are real, and have *all* possible combinations of signs. Then under the conditions given, for all the choices, it is seen that $P$ is a (symmetric) p.d. matrix. With each such $P$ as covariance matrix with mean vector zero, define a $p$-variate normal d.f. (which is unique). Corresponding to $\alpha$, $2\beta$ (integer) and each $P$ so obtained, define a $\phi(t_1, \cdots, t_p)$ by (2). Clearly there are $2^q$ such possible distinct ch.f.'s, since there are that many distinct $P$'s, giving different distributions, all of which by Lemma 1 have the *same* covariance matrix $K$ and, of course, also the same $\alpha$, $\beta$. That is, corresponding to each normal d.f. obtained above we could define a separate multivariate gamma d.f. using (2). This completes the proof.

*Remark.* A gamma distribution unlike the Gaussian is not completely characterized by the covariance matrix $K$ and $\alpha$, $\beta$. Obviously if $k_{ij}=0$ $(i\neq j)$, $q=0$ and the distribution is unique.

**4. Some further considerations.** We need the following

DEFINITION. *A d.f. is said to be infinitely divisible* (*or to have an arithmetical character*) *if it can be written as an n-fold convolution of some d.f. for every n* (*or equivalently, if its ch.f. is the nth power of some ch.f. for every n*).

It is easily seen that such a ch.f. never vanishes, and we use this fact in the sequel. First we have

LEMMA 2. *Suppose* $\mathbf{Z}=(Z_1, \cdots, Z_p)$ *is a correlated sample from the population whose ch.f. is given by* (2). *If* $\mathbf{a}$ *is any nonnull vector* $(1\times p)$, *then* $\mathbf{aZ}'$ *has the d.f. whose ch.f. is* $|I-2it\sigma^2D_aP|^{-\beta}$.

*Proof.* Set $t_j=a_jt$ $(j=1, \cdots, p)$ in (2), where $\mathbf{a}=(a_1, \cdots, a_p)$.

LEMMA 3. *In the above lemma, the d.f. of* $\mathbf{aZ}'$ *holds true if* $\beta>0$ *is any real number, provided that the matrix* $D_aP$ *is at least positive semidefinite* (*p.s.d.*) *and is similar to a real diagonal matrix.*

*Proof.* If $\beta=\frac{1}{2}n$, $n$ an integer, from Lemma 2, the ch.f. of $\mathbf{aZ}'$ is given by

$$\psi(t) = |I - 2it\sigma^2D_aP|^{-\beta}. \tag{4}$$

For convenience let $\alpha=2\sigma^2$ and $B=D_aP$. Let $(c_1, \cdots, c_p)$ be the characteristic roots of $B$. From the hypothesis on $B$, it is seen that all $c_i$ $(i=1, \cdots, p)$ are nonnegative. Then

$$\psi(t) = |I - it\alpha B|^{-\beta} = [\prod (1 - it\alpha c_j)]^{-\beta} = [\prod (1 - itd_j)]^{-\beta},$$

where the product ranges are $j=1, \cdots, p$, and $d_j=\alpha c_j\geqq 0$.

Since $\beta=\frac{1}{2}n$ in Lemma 2, $\psi(t)=[\prod(1-itd_j)]^{-n/2}=[\eta(t)]^n$, say. But $\eta(t)=\prod(1-itd_j)^{-\frac{1}{2}}$ is a ch.f. of the convolution of $p$ gamma distributions, and $\psi(t)$, being the product of $n$ such ch.f.'s, each of which has the arithmetical character [7], is infinitely divisible and hence never vanishes. Therefore, the following operations are meaningful.

From $\eta(t)=\exp\{(1/n)\log\psi(t)\}$, one obtains

$$[\eta(t)]^m = \exp\{(m/n)\log\psi(t)\} \tag{5}$$

is a ch.f., for all integers $m, n>0$. Thus if $\beta$ is rational, (4) is a ch.f. If $\beta$ is real, then there exist sequences of rationals $n/m$ such that limit $n/m=\beta$. Thus letting $n, m\to\infty$, so that their ratio tends to $\beta$, we observe from (5) that this is a sequence of ch.f.'s tending to a limit which is continuous at $t=0$. Hence by the continuity theorem for ch.f.'s [2], the limit is a ch.f. so that $\exp\{\beta^{-1}\log\psi(t)\}$ (hence (4)) is a ch.f., if $\beta$ is any real (positive) number, as was to be proved.

We remark here that in (4) one may be tempted to replace the half-integer by any real positive number. However, that is not correct in general. More precisely, let $\phi(t_1, \cdots, t_p; n_1, \cdots, n_r)$ be a ch.f. for every set of positive integers $n_1, \cdots, n_r$. It does not follow from this that $\phi(t_1, \cdots, t_p; \beta_1, \cdots, \beta_r)$ is a ch.f. for every set of positive real numbers $\beta_1, \cdots, \beta_r$. A simple example* of this is the ch.f. of the binomial distribution

$$\phi(t; n) = (q + pe^{it})^n \qquad (p, q > 0,\ p + q = 1). \tag{6}$$

Another example, which is a multivariate distribution, is the following (given in another context in [4]). This is a slightly restrictive example which nevertheless is of interest. Consider the ch.f. given by ($m, n>0$ integers)

$$(1 - it_1)^{-mn/2}(i - it_2)^{-m/2}\left[1 + \frac{\overset{2}{\rho_{12}}t_1t_2}{(1 - it_1)(1 - it_2)}\right]^{-m/2}. \tag{7}$$

Then a "generalization" is to write this form as

$$(1 - it_1)^{-M}(1 - it_2)^{-N}\left[1 - \frac{\overset{2}{\rho_{12}}t_1t_2}{(1 - it_1)(1 - it_2)}\right]^{-Q} \tag{8}$$

for a joint ch.f. of two positive random variables $X$ and $Y$, where $M, N, Q>0$ are any real numbers. However, on expansion and inversion, one finds that the "frequency function" of (7) being convergent for $|\rho_{12}|<1$ is negative for certain positive values of $x$ and $y$ if $Q\rho_{12}^2>\min(M, N)$, [4], which shows that the convergence condition alone does not ensure the desired conclusion, and consequently for general $Q, M, N$, (7) cannot be a ch.f. But because of Lemma 3, infinite divisibility is a sufficient condition for this "replacement."

* This was pointed out by the referee. We wish to thank the referee for this and various other useful remarks.

If $X_1$ and $X_2$ are independent normal random variables with zero means and unit variances, then Lévy showed, in [6], that the joint distribution of any two of $X_1^2$, $X_1X_2$, and $X_2^2$ is infinitely divisible, but that the joint distribution of the three, which is a Wishart distribution, is not infinitely divisible. In general, for multivariate distributions, infinite divisibility is a nontrivial property to establish ([6], [8]).

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## A SINGLE POSTULATE FOR GROUPS

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Marlow Sholander [1] shows that if we make certain natural presuppositions (*e.g.*, about the structure of English grammar and the definition of an operator) we may characterize an abelian group by a single postulate. This note provides a similar result for an arbitrary group.

Let $G$ be a nonempty set on which two operations are defined: a binary *multiplication* and a singularly *inversion*. For each $a$, $b \in G$ we write $(ab)$ for the product in that order of $a$ and $b$, and $a'$ for the inverse of $a$.

THEOREM. *If* $(ab)c = (ad)f$ *implies* $b = d(fc')$ *for every* $a$, $b$, $c$, $d$, $f \in G$, *then* $G$ *is a group relative to the operations described above.*

*Proof.* 1. Since $(ab)c = (ab)c$ we have $b = b(cc')$. Let us write $a^*$ for $aa'$. Then $(ab^*)c = (ad^*)c$ so that

$$b^* = d^*c^* = d^*.$$

Thus for all $a \in G$, $aa' = e_r$ (a right identity).

2. If $ab=ad$ then $(ab)c=(ad)c$, so that

$$b = d(cc') = d.$$

We thus have left cancellation.

3. We have $(ae_r)b=(ab)e_r$, so that $e_r=b(e_rb')$. But $e_r=bb'$, so that, by left cancellation, $b'=e_rb'$. In particular,

$$e_r' = e_re_r' = e_r.$$

Next, $(ab)e_r=(ae_r)b$, so that $b=e_r(be_r')=e_rb$.

4. We have $(e_rc)c'=(e_rd)d'$ so that $c=d(d'c'')$. If we take $d=e_r$, we find, in particular, that $c=c''$. So

$$c = d(d'c),$$

and the equation $a=bx$ has a solution $x=b'a$.

5.† Given $a$, $b$, $c$ let us choose $d$ so that $[a(bc)]d=(ab)c$. Then $bc=b(cd')$, $ce_r=c=cd'$, $e_r=d'$, and $d=e_r$. We thus have associativity, and the rest is easy.

*Added in proof.* Since writing the above, I have come across an article‡ in which a single axiom for groups is given. The axiom is written in terms of the operator $\delta$, where $ab\delta=ab'$.

More generally, let $W(x_1, \cdots, x_n, \delta)$ be a ("meaingful") word in the symbols $x_1, \cdots, x_n, \delta$. Suppose a class $\mathfrak{A}$ of groups is such that a given group $G$ with unit element $e$ is a member of $\mathfrak{A}$ if and only if $W(a_1, \cdots, a_n, \delta)=e$ for all $a_1, \cdots, a_n \in G$. For example, if $\mathfrak{A}$ is the class of *all* groups, we may take $W=xx\delta$; if $\mathfrak{A}$ is the class of abelian groups, we may take $W=xyyx\delta\delta\delta$.

It is then shown that the class $\mathfrak{A}$ is characterized by the single axiom

$$(\alpha) \qquad xxx\delta W\delta y\delta z\delta x x\delta x\delta z\delta\delta\delta = y,$$

where $W=W(x_1, \cdots, x_n, \delta)$. That is, if $G$ is a set on which a binary operation $\delta$ is defined, then $G$ is a member of $\mathfrak{A}$ if and only if $(\alpha)$ is satisfied for all $x, y, z, x_1, \cdots, x_n \in G$.

### Reference

1. Marlow Sholander, Postulates for commutative groups, this MONTHLY, vol. 66, 1959, pp. 93–95.

† I am indebted to the referee for pointing out a simplification in my treatment of 5.

‡ Graham Higman and B. H. Neumann, Groups as groupoids with one law, Publ. Math. Debrecen, vol. 2, 1952, pp. 215–221.

# A THEOREM IN THE FAREY SERIES

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In this note we shall prove the following

THEOREM. *Let $F_n$ be the Farey series of order $n$ with the end fractions 0 and 1 deleted. Then, for $n \geqq 4$, $f_1 = f_2 f_3$, where $f_i \in F_n$ $(i = 1, 2, 3)$ and $f_2 \neq f_3$, if and only if $f_1$ does not lie among the largest $[\frac{1}{2}(n+1)]$ elements of $F_n$.*

Let $S_n$ be the set of all $f \in F_n$ which are expressible as $f_2 f_3$, where $f_2, f_3 \in F_n$ and $f_2 \neq f_3$. Let $F_n$ be divided into two exhaustive mutually exclusive subsets $R_{n_1}$, $R_{n_2}$ as follows:

$$f_a \in R_{n_1} \text{ if and only if } 0 < f_a \leqq (n-2)/n,$$

$$f_a \in R_{n_2} \text{ if and only if } (n-2)/n < f_a < 1.$$

LEMMA 1. *$f_a \in R_{n_2}$ if and only if $f_a$ is of the form $(x-1)/x$ with $x > \frac{1}{2}n$.*

*Proof.* If $a/b > (n-2)/n$, then $n(b-a) < 2b$ and $b - a < (2b)/n \leqq 2$. Hence $a = b-1$ and $a/b = (b-1)/b$, which is of the form $(x-1)/x$. Also, $(x-1)/x > (n-2)/n$ implies $x > \frac{1}{2}n$.

Conversely, if $x > \frac{1}{2}n$, then $(x-1)/x = 1 - (1/x) > 1 - (2/n) = (n-2)/n$.

LEMMA 2. *The number of elements in $R_{n_2}$ is $[\frac{1}{2}(n+1)]$.*

*Proof.* By Lemma 1, all the elements of $R_{n_2}$ are of the form $(x-1)/x$ with $x > \frac{1}{2}n$, and any element of that form is in $R_{n_2}$. If $n$ is odd, there are $\frac{1}{2}(n+1)$ integers $x$ such that $\frac{1}{2}n < x \leqq n$. Consequently, $\frac{1}{2}(n+1)$ fractions of the form $(x-1)/x$ with $x > \frac{1}{2}n$ can be formed. If $n$ is even, then $\frac{1}{2}n$ such fractions can be formed. In either case the number of such fractions is $[\frac{1}{2}(n+1)]$.

LEMMA 3. *If $f_a \in R_{n_2}$ then $f_a \notin S_n$.*

*Proof.* From Lemma 1 we conclude that the two largest elements of $F_n$ are $(n-1)/n$ and $(n-2)/(n-1)$. Their product, consequently, will be the greatest possible product of two distinct elements of $F_n$, that is, the greatest element of $S_n$. Since the product is equal to $(n-2)/n$ and since $f_a \in R_{n_2} \Rightarrow f_a > (n-2)/n$, no element of $R_{n_2}$ is in $S_n$.

LEMMA 4. *If $f_a \in R_{n_1}$ then $f_a \in S_n$.*

*Proof.* If $f_a \in R_{n_1}$, then $f_a$ cannot be of the form $(x-1)/x$ with $x > \frac{1}{2}n$ (by Lemma 1). Therefore, if $a/b \in R_{n_1}$, either $a < b-1$ or $a = b-1$ and $b \leqq \frac{1}{2}n$.

In the first case, $1 \leqq a < b-1 < b \leqq n$. Therefore $1 \leqq a < a+1 < n$ and $1 < a+1 < b \leqq n$; hence $a/(a+1) \in F_n$ and $(a+1)/b \in F_n$. Since

$$\frac{a}{b} = \left(\frac{a}{a+1}\right)\left(\frac{a+1}{b}\right),$$

$a/b \in S_n$ in the first case.

In the second case, $1<2b-1<2b\leqq n$ and $1<2b-2<2b-1<n$; hence $(2b-1)/(2b)\in F_n$ and $(2b-2)/(2b-1)\in F_n$. Since

$$\frac{a}{b}=\frac{b-1}{b}=\left(\frac{2b-1}{2b}\right)\left(\frac{2b-2}{2b-1}\right),$$

$a/b\in S_n$ in the second case also.

*Proof of the theorem.* In Lemmas 1–4 we have proved that all elements of $R_{n_1}$ are in $S_n$, that no element of $R_{n_2}$ is in $S_n$, and that the number of elements of $R_{n_3}$ is $[\frac{1}{2}(n+1)]$. Therefore the proof of the theorem is complete.

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## THE ERDÖS INEQUALITY AND OTHER INEQUALITIES FOR A TRIANGLE

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*Note added in proof* (February, 1961).* From a copy of the paper by A. Florian kindly sent to me by the author, I learn that the reviewer in *Mathematical Reviews* has misquoted the results for $k<0$. Florian's results agree with mine.

Through the kindness of Professor H. S. M. Coxeter and the erudition of Professor E. H. Neville, I learn that the triangle theorem of Section 9 is not new. It is stated in the last edition of Casey's *A Sequel to Euclid*, 1892, page 253, and is apparently due to Neuberg. I find it remarkable that an elementary theorem of such charm should disappear from view.

My extension to plane polygons appears to be new.

* A delay in transit prevented this note from being added to the original article, this MONTHLY, vol. 68, 1961, pp. 226–230.

### CORRECTION

It has been pointed out by Donald W. Western that Theorem 1 of his paper *Inequalities of the Markoff and Bernstein type for integral norms*, Duke Math. J., vol. 15, 1948, pp. 839–869, contains as a special case Theorem 1 of the paper by Q. I. Rahman, *Some inequalities for polynomials*, this MONTHLY, vol. 67, 1960, pp. 847–851. This fact was not known to Mr. Rahman nor to the referee.

## MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

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### RATIONAL ORTHOGONAL MATRICES

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The construction of a $3\times3$ rotation matrix $R$ over the rationals is equivalent to the construction of a similarity matrix $S$ whose rows are mutually orthogonal vectors $\alpha$, $\beta$, and $\gamma$ which have integral components and share an integral length $D$. Some textbooks give the impression the construction is essentially unique with $\alpha=[1, 2, 2]$, $\beta=[2, 1, -2]$, and $\gamma=[2, -2, 1]$.

It is well known that a rotation may be obtained as a product of rotations about the $z$, $y$, and $z$ axes, respectively, and that $R$ may be expressed as the Eulerian matrix

$$\begin{pmatrix} c_1c_2c_3 - s_1s_3 & c_1c_2s_3 + s_1c_3 & -c_1s_2 \\ -s_1c_2c_3 - c_1s_3 & -s_1c_2s_3 + c_1c_3 & s_1s_2 \\ s_2c_3 & s_2s_3 & c_2 \end{pmatrix}$$

where $c_i=\cos\theta_i$ and $s_i=\sin\theta_i$ for $i=1, 2, 3$. We note that to have these elements rational it is not necessary that all $c_i$ and $s_i$ be rational. Suppose that we have found integers satisfying

$$(*) \qquad \alpha_3^2+\beta_3^2+\gamma_3^2=D^2=\gamma_1^2+\gamma_2^2+\gamma_3^2.$$

We construct $R$ by choosing $\cos\theta_2=\gamma_3/D$, $\tan\theta_1=\beta_3/\alpha_3$, and $\tan\theta_3=\gamma_2/\gamma_1$.

The construction of $S$ now is reduced to finding integral solutions of (*). This Diophantine problem is discussed in L. E. Dickson's *History of the Theory of Numbers*, vol. II, Washington, 1919, pp. 261–269. Elegant parametric representations for classes of matrices $S$, based on identities due to Lebesgue, Catalan, and Dainelli (loc. cit.) are given below.

First, let

$$\alpha = [a^2+b^2-c^2-d^2,\ 2(ac+bd),\ 2(ad-bc)],$$
$$\beta = [2(ac-bd),\ b^2+c^2-a^2-d^2,\ 2(ab+cd)],$$
$$\gamma = [2(ad+bc),\ 2(cd-ab),\ b^2+d^2-a^2-c^2].$$

Here $D=a^2+b^2+c^2+d^2$. When $d$ is chosen as shown below and a factor 2 is removed, we get the following special cases. Where $d=a+b+c$, let $\alpha=[-ab-cd,\ ac+bd,\ ad-bc]$, $\beta=[ac-bd,\ -bc-ad,\ ab+cd]$, and $\gamma=[bc+ad,\ cd-ab,\ ac+bd]$. Here $D=a^2+b^2+c^2+ab+bc+ca$. If we take $c=0$, we obtain the next case. Where $d=a+b$, let $\alpha=[-ab, bd, ad]$, $\beta=[-bd, -ad, ab]$, and $\gamma=[ad, -ab, bd]$. Here $D=a^2+ab+b^2$.

## REMARK ON THE CONSTRAINT SETS IN LINEAR PROGRAMMING

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In the standard linear programming problem, one is given a real $m \times n$ matrix $A$ and vectors $b \in \mathfrak{R}^m$ and $c \in \mathfrak{R}^n$ where $\mathfrak{R}^k$ is euclidean $k$-space. One forms the *primal constraint set*, here designated by $\mathfrak{F}(A, b) = \{x \mid Ax \leqq b, 0 \leqq x\}$, and the *dual constraint set* $\mathfrak{F}(c, A) = \{u \mid c^T \leqq u^T A, 0 \leqq u\}$. The primal problem is to maximize $c^T x$ over all $x \in \mathfrak{F}(A, b)$ and the dual problem is to minimize $u^T b$ over all $u \in \mathfrak{F}(c, A)$. The "existence theorem" [3; p. 61] states that either problem has a solution (and hence both do) if and only if neither constraint set is empty. It appears not to have been noticed that it is impossible for both constraint sets to be bounded.

THEOREM. *If a linear programming problem has a solution, then either the primal constraint set or the dual constraint set is unbounded.*

*Proof.* By [2; p. 49], if $\mathfrak{F}(A, b)$ is not empty, then it is bounded if and only if $\mathfrak{F}(A, 0) = \{0\}$. Similarly if $\mathfrak{F}(c, A)$ is not empty, then it is bounded if and only if $\mathfrak{F}(0, A) = \{0\}$. The assumption that the linear programming problem has a solution implies that $b$ and $c$ have been so chosen that neither constraint set is empty. Beyond this condition no further use is made of $b$ and $c$; we prove that it is impossible to have both $\mathfrak{F}(A, 0) = \{0\}$ and $\mathfrak{F}(0, A) = \{0\}$.

Now $\mathfrak{F}(A, 0) = \{x \mid Ax \leqq 0, 0 \leqq x\}$, a pointed, convex polyhedral cone. Its polar cone is $\mathfrak{F}^*(A, 0) = \{y \mid x \in \mathfrak{F}(A, 0) \Rightarrow y^T x \leqq 0\}$, the set of all vectors in $\mathfrak{R}^n$ which do not form an acute angle with any vector of $\mathfrak{F}(A, 0)$. Thus $\mathfrak{F}(A, 0) = \{0\}$ if and only if $\mathfrak{F}^*(A, 0) = \{\text{all } y \in \mathfrak{R}^n\}$. But by the theorem of Farkas [1; p. 31], $\mathfrak{F}^*(A, 0) = \{y \mid y = A^T u - v, 0 \leqq u, 0 \leqq v\} = \{y \mid y^T \leqq u^T A, 0 \leqq u\}$. Select any $y \in \mathfrak{R}^n$ with $0 < y$. Since $y \in \mathfrak{F}^*(A, 0)$, there exists a nonzero $u \in \mathfrak{R}^m$ with $y^T \leqq u^T A, 0 \leqq u$.

One may reason in a similar fashion for $\mathfrak{F}(0, A)$. Instead one may simply observe that $\mathfrak{F}(0, A) = \mathfrak{F}(-A^T, 0)$. Hence $\mathfrak{F}^*(0, A) = \{v \mid v^T \leqq x^T(-A^T), 0 \leqq x\}$ and this set must consist of all $v \in \mathfrak{R}^m$ if $\mathfrak{F}(0, A) = \{0\}$. Select any $v \in \mathfrak{R}^m$ with $0 < v$; then there exists a nonzero $x \in \mathfrak{R}^n$ with $Ax \leqq -v, 0 \leqq x$. Now if both $\mathfrak{F}(A, 0) = \{0\}$ and $\mathfrak{F}(0, A) = \{0\}$ it would follow that $0 < y^T x \leqq u^T A x \leqq -u^T v < 0$, a contradiction.

One may also prove the theorem more briefly but less directly by showing that if the theorem were false, this would deny the property of complementary slackness as given in [1; Theorem 6]. With the appropriate changes in notation, the theorem referred to states (in part) that there must be some solution of the inequalities $0 \leqq u$, $0 \leqq A^T u$, $Ax \leqq 0$, $0 \leqq x$ for which $Ax < u$ and $-A^T u < x$; thus $x = 0$, $u = 0$ cannot be the only solution of this system of inequalities.

### References

1. A. J. Goldman and A. W. Tucker, Polyhedral convex cones, in H. W. Kuhn ed. Linear inequalities and related systems, Princeton, 1956, pp. 19–40. [Annals of Mathematics Studies no. 38].

2. A. J. Goldman, Resolution and separation theorems for polyhedral convex sets, in H. W. Kuhn ed. Linear inequalities and related systems, Princeton, 1956, pp. 41–51. [Annals of Mathematics Studies no. 38.]

3. A. J. Goldman and A. W. Tucker, Theory of linear programming, in H. W. Kuhn ed. Linear inequalities and related systems, Princeton, 1956, pp. 53–97. [Annals of Mathematics Studies no. 38].

## IRREDUCIBILITY OF CERTAIN BERNOULLI POLYNOMIALS*

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The Bernoulli polynomials are defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

$B_m(x)$ is a polynomial of degree $m$ having rational coefficients. Its constant term, $B_m = B_m(0)$, is the $m$th Bernoulli number in Nörlund's notation [2, Chapter 2]. We have [2, p. 18]

$$B_m(x) = \sum_{r=0}^{m} \binom{m}{r} B_r x^{m-r}.$$

We are interested in determining values of $m$ for which $B_m(x)$ is irreducible (over the field of rational numbers). When $m$ is odd, $B_m(x)$ is divisible by $x(x-1)(x-1/2)$, so we shall confine our attention to even values of $m$. The only known results are those of Carlitz [1] that $B_m(x)$ is irreducible whenever $m = k(p-1)p^t$, $p$ a prime, $t \geqq 0$, and $1 \leqq k \leqq p$.

Carlitz obtained his results by showing that for the values of $m$ mentioned above, $pB_m(x)$ is the Eisenstein polynomial, *i.e.*, it satisfies the hypotheses of the Eisenstein irreducibility criterion, with $p$ as the determining prime. There are other cases when this is true and we shall determine all of them here. The leading coefficient of $pB_m(x)$ is $p$ while its constant term is $pB_m$. Hence it follows from the Staudt-Clausen theorem [2, p. 33] that if $pB_m(x)$ is to be an Eisenstein polynomial, with $p$ as the determining prime, we must have $m = k(p-1)$. The coefficient of $x^{m-r}$ in $pB_m(x)$ is then

$$\binom{k(p-1)}{r} pB_r. \tag{1}$$

Suppose that $0 < r < k(p-1)$. If $p-1$ does not divide $r$, it follows from the Staudt-Clausen theorem that $pB_r \equiv 0 \pmod{p}$, so that (1) is divisible by $p$. If $p-1$ does divide $r$, then $pB_r \equiv -1 \pmod{p}$, again from the Staudt-Clausen theorem. Hence $pB_m(x)$ is an Eisenstein polynomial, with $p$ as the determining prime, if and only if the binomial coefficient in (1) is divisible by $p$ for all $r$ such that $0 < r < k(p-1)$ and $p-1$ divides $r$. A necessary and sufficient condition for

* This work was supported by NSF-G11293.

this to be true has been given by Schäffer [3, Lemma 3]. If $n=a_0+a_1p+a_2p^2+\cdots$, where $0\leqq a_i\leqq p-1$ for $i=0, 1, \cdots$, we set $A(n, p)=a_0+a_1+\cdots$. Schäffer's lemma is then the following:

*If $p$ is a prime and $n$ is an arbitrary positive integer, then*

$$\binom{n}{s(p-1)} \equiv 0 \pmod{p}$$

*for all integers $s$ such that $0<s(p-1)<n$, if and only if $A(n, p)\leqq p-1$.*

We therefore have the following result.

THEOREM. *$pB_m(x)$ is an Eisenstein polynomial, with $p$ as the determining prime, if and only if $p-1$ divides $m$ and $A(m, p)\leqq p-1$.*

Using this we can state several special results. We give only one example: *if $k+1\leqq p$ then $B_{(kp+k+1)(p-1)}(x)$ is irreducible.*

If $m$ is even and $0\leqq m\leqq 200$, the result of Carlitz tells us that $B_m(x)$ is irreducible except for 25 values of $m$. Of these 25 cases application of the above theorem shows that $B_{76}(x)$, $B_{114}(x)$, $B_{152}(x)$, and $B_{170}(x)$ are irreducible. Of the remaining 21 cases, $m$ is twice a prime in 15 cases. This points up the desirability of showing that $B_{2p}(x)$ is irreducible when $p$ is a prime.

**References**

1. L. Carlitz, Note on irreducibility of the Bernoulli and Euler polynomials, Duke Math. J., vol. 19, 1952, pp. 475–481.

2. N. E. Nörlund, Vorlesungen über Differenzenrechnung, Berlin, 1924.

3. Juan J. Schäffer, The equation $1^p+2^p+3^p+\cdots+n^p=m^q$, Acta Math., vol. 95, 1956, pp. 155–189.

## THE CONVERGENCE OF SEQUENCES WITH LINEAR FRACTIONAL RECURRENCE RELATION

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For the sake of brevity, and by analogy with [1], we make the

DEFINITION. *A (complex-valued) sequence with initial $z_1$ and recurrence relation*

$$z_{n+1}=\frac{az_n+b}{z_n+d}, \qquad ad-b\neq 0 \tag{1}$$

*is called the Moebius sequence $M(z_1; a, b, d)$.*

In this note we shall give a formula for the $n$th term of a Moebius sequence. From this, convergence can readily be discussed. Finally, we obtain a necessary and sufficient condition for a Moebius sequence to take only a finite number of values.

To include the case where $z_n=-d$ for some $n$, we add the point $z=\infty$ to the

complex plane. $M(z_1; a, b, d)$ has a member $z_r = \infty$ (*i.e.*, $z_{r-1} = -d$) if and only if $z_1$ is a member of $M(-d; -d, b, -a)$.

As usual, the sequence $\{z_n\}$ is said to be convergent if and only if $\lim_{n\to\infty} z_n$ exists and is not infinite.

Rewriting (1) in the form

$$z_{n+1}z_n + dz_{n+1} - az_n - b = 0, \tag{1*}$$

we form the "auxiliary equation"

$$z^2 + (d - a)z - b = 0 \tag{2}$$

whose roots, $\alpha$ and $\beta$, are the only values to which $z_n$ can converge.

$$\alpha + \beta = a - d; \qquad \alpha\beta = -b. \tag{3}$$

*Remarks.*

(i) If $\alpha$ is a root of (2) and if $z_n = \alpha$ for some value of $n$, then $z_n = \alpha$ for all $n$, *i.e.* the sequence is single valued.

(ii) If the sequence is not single valued, then $z_{n+1} \neq z_n$ for $n = 1, 2, \cdots$.

In general, two cases arise:

(a) $\alpha \neq \beta$. From (1*) and (3) and the remarks above we obtain, for any $n$,

$$\frac{z_{n+1} - \alpha}{z_{n+1} - \beta} = \lambda \frac{z_n - \alpha}{z_n - \beta}, \quad \lambda = \frac{d + \beta}{d + \alpha}. \tag{4}$$

$\lambda \neq 0$ or $\infty$, since if $d = -\beta$ or $-\alpha$ then $ad - b = 0$, and the recurrence relation (1) collapses.

(b) $\alpha = \beta$. In this case

$$\frac{1}{z_{n+1} - \alpha} = \frac{1}{z_n - \alpha} + \mu, \quad \mu = \frac{1}{d + \alpha}. \tag{5}$$

Repeated application of (4) and (5) gives

THEOREM 1. *The Moebius sequence $M(z_1; a, b, d)$ has nth term*

$$z_n = \frac{\alpha(z_1 - \beta) - \lambda^{n-1}\beta(z_1 - \alpha)}{(z_1 - \beta) - \lambda^{n-1}(z_1 - \alpha)}, \qquad \alpha \neq \beta, \qquad \lambda = \frac{d + \beta}{d + \alpha};$$

$$z_n = \frac{(z_1 - \alpha)(d + \alpha)}{(n - 1)(z_1 - \alpha) + (d + \alpha)} + \alpha, \quad \alpha = \beta, \tag{6}$$

*where $\alpha$ and $\beta$ are the roots of the auxiliary equation* (2).

Concerning the convergence of Moebius sequences:

THEOREM 2. *$M(z_1; a, b, d)$ is single valued if $z_1 = \alpha$ or $\beta$. For other initial values, the sequence*

(*a*) *converges to $\alpha$, if $\alpha = \beta$ or $|d+\alpha| > |d+\beta|$,*

(*b*) *diverges if $|d+\alpha| = |d+\beta|$, $\alpha \neq \beta$.*

*Proof.* (*a*) follows from (6). As for (*b*), let $\lambda = e^{i\theta}$, $\theta \not\equiv 0 \pmod{2\pi}$. Then, from (4),

$$\frac{z_n - \alpha}{z_n - \beta} = e^{i(n-1)\theta} \frac{z_1 - \alpha}{z_1 - \beta}. \tag{7}$$

Thus $z_n$ cannot converge to $\alpha$ or $\beta$. We note that by suitable selection of the root $\alpha$, one of the cases (*a*), (*b*) must occur.

DEFINITION. *A sequence* $\{z_n\}$ *is cyclic of period* $m$ *if* $m$ *is the least integer such that, for all* $n$, $z_{n+m} = z_n$.

THEOREM 3. *The Moebius sequence* $M(z_1; a, b, d)$, $z \neq \alpha$ *or* $\beta$, *is cyclic of period* $m > 1$ *if and only if there is an integer* $k$, $(k, m) = 1$, *for which*

$$\frac{(a+d)^2}{(a-d)^2 + 4b} = -\cot^2 \frac{k\pi}{m}. \tag{8}$$

*Proof.* From (7), $z_{m+1} = z_1$ and $z_n \neq z_1$ for $1 < n \leqq m$ if and only if $m\theta = 2k\pi$, for some integer $k$ which is relatively prime to $m$. But

$$e^{2k\pi i/m} = e^{i\theta} = \lambda = \frac{d+\beta}{d+\alpha} = \frac{(a+d) \pm \sqrt{[(a-d)^2 + 4b]}}{(a+d) \mp \sqrt{[(a-d)^2 + 4b]}}$$

from which the result follows.

Because of its frequent application, we give the following special case of Theorem 2 as

THEOREM 2R. *The real Moebius sequence* $M(x_1; a, b, d)$, $x_1, a, b, d$ *real, whose auxiliary equation* (2) *has real roots* $\alpha$ *and* $\beta$, $\alpha \geqq \beta$, *is single valued if* $x_1 = \alpha$ *or* $\beta$. *For other initial values the sequence*

(*a*) *converges to* $\alpha$, *if* $\alpha = \beta$ *or* $a + d > 0$, *and converges to* $\beta$ *if* $a + d < 0$

(*b*) *is cyclic of period* 2 *if* $a + d = 0$.

*Proof.* Since $a \geqq \beta$, $2(d+\alpha) = (a+d) + \sqrt{[(a-d)^2 + 4b]}$ and $2(d+\beta) = (a+d) - \sqrt{[(a-d)^2 + 4b]}$. Thus $|d+\alpha| > |d+\beta|$ implies that $a+d > 0$. The case $a+d=0$, $\alpha = \beta$ cannot arise, for this would imply that $ad = b$. If $a+d=0$, $\alpha \neq \beta$, then the denominator of (8) is not zero, and Theorem 3 applies.

The only real Moebius sequences not covered in Theorems 3 and 2*R* are those which are not cyclic and for which $\alpha$ and $\beta$ are not real. Clearly such sequences cannot converge, and, as required by Theorem 2, $|d+\alpha| = |d+\beta|$.

**Reference**

1. C. Carathéodory, Conformal Representation, Cambridge Tracts in Mathematics and Mathematical Physics, No. 28. Cambridge, 1932.

## THE RECIPROCAL ITERATED LIMIT THEOREM

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Let $I$ and $J_i$, $i\in I$, be directed sets (for nomenclature see Kelley, *General Topology*, Ch. II), and let $S=\{s_{ij}: i\in I, j\in J_i\}$ be a double net in a topological space $T$, which is simply convergent: $r_i=\lim_{j\in J_i} s_{ij}$. Let $P$ be the product directed set $I\times\prod_{i\in I} J_i$, and if $p=\{i, \{j_k: k\in I\}\}\in P$, let $s_p=s_{ij_i}$. The net $\{s_p: p\in P\}$ is called the diagonal net of $S$, and an important elementary theorem of general topology states:

ITERATED LIMIT THEOREM. *If the net $\{r_i: i\in I\}$ (obtained from the limits of a simply convergent double net) is convergent, then its diagonal net converges to the same limit.*

Here we wish to prove the following elementary reciprocal theorem:

THEOREM. *A topological space $T$ is regular if and only if, for every simply convergent double net $S$ it is true that $\lim_{p\in P} s_p=r$ implies $\lim_{i\in I} r_i=r$ (notation as above).*

*Proof.* 1) Let $T$ be regular and $\lim s_p=r$, and suppose that $r_i$ does not converge to $r$. Then there are a closed neighborhood $V$ of $r$ and a subnet of $r_i$ which do not meet.

Since $\lim_{p\in P} s_p=r$, there is a $p^0=\{i^0, \{j_k^0: k\in I\}\}$ such that $p\geqq p^0$ implies $s_p\in V$. Then for each $i\geqq i^0$ and $j\geqq j_i^0$, put $p'=\{i, \{j_k': k\in I\}\}$, with $j_i'=j$ and $j_k'=j_k^0$ for $k\neq i$. Then $p'\geqq p^0$, and $s_{ij}=s_{p'}\in V$. Let $i\geqq i^0$ and such that $r_i\notin V$, and let $W$ be a neighborhood of $r_i$ disjoint from $V$ ($V$ is closed). Since $s_{ij}\to r_i$, there is a $j^0\in J_i$ such that $j\geqq j^0$ implies $s_{ij}\in W$. Take $j\geqq j^0$, $j_i^0$ and we have a contradiction.

2) Suppose that $T$ is not regular, that is, there is a point $r$ in $T$ and a neighborhood $V$ of $r$ such that for each neighborhood $W$ of $r$ the set difference $\overline{W}\sim V$ is not empty. Take $r_W\in\overline{W}\sim V$, and, for each $W$ and each neighborhood $U$ of $r_W$ take $s_{WU}\in U\cap W$. Call $I(J_W)$ the directed set of all neighborhoods of $r(r_W)$. Then obviously $\lim_{U\in J_W} s_{WU}=r_W$ and the diagonal net $s_p=s_{WU_W}$, with $p=\{W, \{U_S: S\in I\}\}$, converges to $r$, since for each $W^0\in I$, $W\subset W^0$ implies $s_{WU_W}\in W^0$. Yet of course $r_W$ does not converge to $r$.

## NOTE ON THE DIVISORS OF A NUMBER

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Let $\nu(n)$ and $\tau(n)$ denote the number of distinct prime divisors and the number of divisors of $n$ respectively. Define $\beta(n)=\sum_{d/n}\nu(d)$ and $\alpha(n)=\beta(n)/\tau(n)$. Thus $\alpha(n)$ is the average number of distinct prime divisors of the divisors of $n$. The object of this note is to derive a simple expression for $\alpha(n)$ and to calculate the average order of $\alpha(n)$. In particular, it is shown that $\alpha(n)$ has average order $\frac{1}{2}\nu(n)$.

THEOREM 1. $\beta(n)=\sum_{p|n}\tau(n/p)$, *where the sum extends over the distinct prime divisors of* $n$.

*Proof.* For $s>1$ we have

$$\sum_{n=1}^{\infty}\frac{\beta(n)}{n^s}=\zeta(s)\sum_{n=1}^{\infty}\frac{\nu(n)}{n^s}=\zeta^2(s)\sum_{p}\frac{1}{p^s}$$

$$=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}\sum_{n=1}^{\infty}\frac{x(n)}{n^s}=\sum_{n=1}^{\infty}\frac{\gamma_n}{n^s},$$

where $x(n)=1$ if $n$ is a prime and zero otherwise and

$$\gamma_n=\sum_{d|n}x(d)\tau\left(\frac{n}{d}\right)=\sum_{p|n}\tau\left(\frac{n}{p}\right).$$

The result now follows from the uniqueness theorem for Dirichlet series.

THEOREM 2. $\alpha(n)=\sum_{k=1}^{r}\alpha_k/\alpha_k+1$, *where* $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ *is the representation of* $n$ *as a product of powers of distinct primes.*

*Proof.* Since $\tau(n)=(\alpha_1+1)\cdots(\alpha_r+1)$, we have

$$\tau\left(\frac{n}{p_k}\right)=\frac{\alpha_k}{\alpha_k+1}\tau(n).\quad\text{Hence}\quad\beta(n)=\tau(n)\sum_{k=1}^{r}\frac{\alpha_k}{\alpha_k+1}$$

by Theorem 1 and the desired result follows from the definition of $\alpha(n)$. The following corollary is now obvious.

COROLLARY. $\frac{1}{2}\nu(n)\leqq\alpha(n)<\nu(n)$ *and these inequalities are the best possible. In particular,* $\alpha(n)=\frac{1}{2}\nu(n)$ *if and only if* $n$ *is square-free.*

THEOREM 3. $\sum_{n\leq x}\alpha(n)=\frac{1}{2}\sum_{n\leq x}\nu(n)+cx+O(x/\log x)$.

*Proof.* By Theorem 1 we have

$$\sum_{n\leq x}\alpha(n)=\sum_{n\leq x}\sum_{p|n}\frac{\tau(n/p)}{\tau(n)}=\sum_{p\leq x}\sum_{\substack{n\leq x\\ p|n}}\frac{\tau(n/p)}{\tau(n)}$$

$$=\sum_{p\leq x}\sum_{k=1}^{\infty}\frac{k}{k+1}\left(\left[\frac{x}{p^k}\right]-\left[\frac{x}{p^{k+1}}\right]\right)$$

$$=\frac{1}{2}\sum_{p\leq x}\left[\frac{x}{p}\right]+\sum_{p\leq x}\sum_{k=2}^{\infty}\frac{1}{k(k+1)}\left[\frac{x}{p^k}\right]$$

$$=\frac{1}{2}\sum_{n\leq x}\nu(n)+x\sum_{p\leq x}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}+O(\pi(x)).$$

But $O(\pi(x))=O(x/\log x)$ by Tchebycheff's inequality. Also,

$$\sum_{p\le x}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}=\sum_{p}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}-\sum_{p>x}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}$$

$=c+O(1/x)$ and this proves the theorem.

The following corollaries are a consequence of the well-known asymptotic formula

$$\sum_{n\le x}\nu(n)=x\log\log x+c_1x+O\left(\frac{x}{\log x}\right).$$

COROLLARY. $\sum_{n\le x}\alpha(n)=\frac{1}{2}x\log\log x+c_2x+O(x/\log x)$.

COROLLARY. *The average order of* $\alpha(n)$ *is* $\frac{1}{2}\nu(n)$.

The constant $c$ of the preceding theorem can be put in a more suitable form by observing that

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{1}{k(k+1)p^k}&=\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)\frac{1}{p^k}\\&=\sum_{k=2}^{\infty}\frac{1}{kp^k}-p\sum_{k=3}^{\infty}\frac{1}{kp^k}\\&=\frac{1}{2p^2}+(p-1)\left(\frac{1}{p}+\frac{1}{2p^2}\right)-(p-1)\sum_{k=1}^{\infty}\frac{1}{kp^k}\\&=1-\frac{1}{2p}+(p-1)\log\left(1-\frac{1}{p}\right).\end{aligned}$$

Thus

$$c=\sum_{p}\left\{1-\frac{1}{2p}+(p-1)\log\left(1-\frac{1}{p}\right)\right\}.$$

We recall that the normal order of $f(n)$ is $g(n)$ if $(1-\epsilon)g(n)<f(n)<(1+\epsilon)g(n)$ for every $\epsilon>0$ and almost all $n$, *i.e.*, if the number of $n\leqq x$ which do not satisfy these inequalities is $O(x)$.

THEOREM 4. *The normal order of* $\alpha(n)$ *is* $\frac{1}{2}\nu(n)$.

*Proof.* Suppose that $\alpha(n)\geqq(1+\epsilon)\frac{1}{2}\nu(n)$, where $\epsilon>0$. Then, by Theorem 2,

$$\begin{aligned}2\alpha(n)-\nu(n)&=\sum_{k=1}^{r}\left(\frac{2\alpha_k}{\alpha_k+1}-1\right)=\sum_{k=1}^{r}\frac{\alpha_k-1}{\alpha_k+1}\\&=\frac{1}{\tau(n)}\sum_{p^2\mid n}\tau\left(\frac{n}{p^2}\right)\geqq\epsilon\nu(n).\end{aligned}$$

Thus

$$\sum_{p^2|n} \tau\left(\frac{n}{p^2}\right) \geqq \epsilon\nu(n)\tau(n),$$

$$\tau(n) \sum_{p^2|n} 1 \geqq \sum_{p^2|n} \left(\frac{n}{p^2}\right) \geqq \epsilon\nu(n)\tau(n), \qquad i.e., \quad \sum_{p^2|n} 1 \geqq \epsilon\nu(n).$$

But $\Omega(n)-\nu(n) \geqq \sum_{p^2|n} 1$, so that $\Omega(n) \geqq (1+\epsilon)\nu(n)$, where $\Omega(n) = \sum_{k=1}^{r} \alpha_k$ is the total number of prime divisors of $n$.

It is known that $\Omega(n)$ and $\nu(n)$ both have normal order $\log\log n$ and it follows that the number of $n \leqq x$ for which $\Omega(n) \geqq (1+\epsilon)\nu(n)$ is $O(x)$. Also, $\alpha(n) > (1-\epsilon)\frac{1}{2}\nu(n)$ for all $n$. Hence,

$$(1-\epsilon)\tfrac{1}{2}\nu(n) < \alpha(n) < (1+\epsilon)\tfrac{1}{2}\nu(n)$$

for almost all $n$ and this proves the theorem.

**Reference**

G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1954.

## SOME REMARKS ABOUT THE CURL OF A VECTOR FIELD

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If one defines the curl of a differentiable vector field in the usual manner of elementary vector analysis (in terms of partial derivatives with respect to Cartesian coordinates), it is necessary to show that the defining expression is invariant under right-handed orthogonal transformations. It is not difficult to do this, without using Stoke's integration theorem or the machinery of tensor calculus; but if one starts in a straightforward way, the ensuing manipulations are apt to seem tediously complicated. The desire for a simple demonstration led to the following considerations, in which the notion of curl is generalized in an elementary way.

Let $\mathbf{R}^n$ be Euclidean $n$-space, consisting of all sequences of $n$ real numbers, with the usual vector operations and with inner multiplication defined by

$$\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{n} x_i y_i,$$

where $\mathbf{x}=(x_1, \cdots, x_n)$ and $\mathbf{y}=(y_1, \cdots, y_n)$. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be the fundamental unit vectors, so that $x_i = \mathbf{x}\cdot\mathbf{e}_i$ and $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$. Suppose that $\mathbf{R}^n$ is made into an algebra by introducing a binary operation $\wedge$ which is linear but not necessarily associative (this can be done by setting up a "multiplication table" specifying the $n^2$ vectors $\mathbf{e}_i \wedge \mathbf{e}_j$).

A differentiable vector field, defined on an open set $G$ in $\mathbf{R}^n$, is a function $\mathbf{v}$ which maps $G$ into $\mathbf{R}^n$ and has continuous partial derivatives $\partial\mathbf{v}/\partial x_i$. Given

such a function, we can define a vector field $\nabla \wedge \mathbf{v}$ by the formula

$$\nabla \wedge \mathbf{v} = \sum_{i=1}^{n} \mathbf{e}_i \wedge \frac{\partial \mathbf{v}}{\partial x_i} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_i} \mathbf{e}_i \wedge \mathbf{e}_j, \tag{1}$$

where $v_j = \mathbf{v} \cdot \mathbf{e}_j$ (so that $\mathbf{v} = \sum_{j=1}^{n} v_j \mathbf{e}_j$). Now an orthogonal transformation is characterized by the vectors, say $\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_n$, to which $\mathbf{e}_1, \cdots, \mathbf{e}_n$ are respectively transformed; these are subject only to the conditions $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$, and if we write $\hat{x}_j = \mathbf{x} \cdot \hat{\mathbf{e}}_j$ we can define the transformation by the identity

$$\sum_{i=1}^{n} x_i \mathbf{e}_i \equiv \sum_{j=1}^{n} \hat{x}_j \hat{\mathbf{e}}_j. \tag{2}$$

Thus

$$\frac{\partial \hat{x}_j}{\partial x_i} = \hat{\mathbf{e}}_j \cdot \mathbf{e}_i,$$

so that, by the chain rule for partial derivatives, and the linearity of $\wedge$,

$$\nabla \wedge \mathbf{v} = \sum_{i=1}^{n} \mathbf{e}_i \wedge \sum_{j=1}^{n} \hat{\mathbf{e}}_j \cdot \mathbf{e}_i \frac{\partial \mathbf{v}}{\partial \hat{x}_j} = \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} \hat{\mathbf{e}}_j \cdot \mathbf{e}_i \mathbf{e}_i \right\} \wedge \frac{\partial \mathbf{v}}{\partial \hat{x}_j}.$$

But $\sum_{i=1}^{n} \hat{\mathbf{e}}_j \cdot \mathbf{e}_i \mathbf{e}_i = \hat{\mathbf{e}}_j$, so that

$$\nabla \wedge \mathbf{v} = \sum_{j=1}^{n} \hat{\mathbf{e}}_j \wedge \frac{\partial \mathbf{v}}{\partial \hat{x}_j} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial \hat{v}_k}{\partial \hat{x}_j} \hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k, \tag{3}$$

where $\hat{v}_k = \mathbf{v} \cdot \hat{\mathbf{e}}_k$. Comparing (1) and (3), we see that the vector field $\nabla \wedge \mathbf{v}$ is an orthogonal invariant of $\mathbf{v}$. Let us say that $\mathbf{v}$ is "irrotational," in a given domain, if $\nabla \wedge \mathbf{v} = \mathbf{0}$ throughout the domain.

When $n = 1$ and $\wedge$ means ordinary multiplication of real numbers, $\nabla \wedge \mathbf{v}$ is the ordinary derivative of $\mathbf{v}$ and the irrotational fields are the constants. When $n = 2$, $\mathbf{R}^n$ consists of the complex numbers, and we can take $\wedge$ to mean the usual multiplication of these ($\mathbf{e}_1 \wedge \mathbf{e}_1 = \mathbf{e}_1$, $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_2 \wedge \mathbf{e}_1 = \mathbf{e}_2$, $\mathbf{e}_2 \wedge \mathbf{e}_2 = -\mathbf{e}_1$); then we find from (1) that

$$\nabla \wedge \mathbf{v} = \left( \frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2} \right) \mathbf{e}_1 + \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) \mathbf{e}_2.$$

In this case, therefore, $\mathbf{v}$ is irrotational if and only if it satisfies the Cauchy-Riemann equations,

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} \quad \text{and} \quad \frac{\partial v_2}{\partial x_1} = -\frac{\partial v_1}{\partial x_2}.$$

Thus the irrotational 2-vector fields are precisely the holomorphic functions.

When $n = 3$ and $\wedge$ means the usual vector multiplication (defined by the

skewness property $\mathbf{x}\wedge\mathbf{y}=-\mathbf{y}\wedge\mathbf{x}$ together with the cyclic rule $\mathbf{e}_1\wedge\mathbf{e}_2=\mathbf{e}_3$ etc.), we see from (1) that

$$\nabla\wedge\mathbf{v}=\left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1+\cdots=\operatorname{curl}\mathbf{v}.$$

But, since $\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j=\delta_{ij}$, we have $\hat{\mathbf{e}}_1\wedge\hat{\mathbf{e}}_2=\epsilon\hat{\mathbf{e}}_3$ etc., where $\epsilon=\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2\wedge\hat{\mathbf{e}}_3=\pm1$; hence, by (3),

$$\epsilon\nabla\wedge\mathbf{v}=\left(\frac{\partial\hat{v}_3}{\partial\hat{x}_2}-\frac{\partial\hat{v}_2}{\partial\hat{x}_3}\right)\hat{\mathbf{e}}_1+\cdots.$$

Now the transformation defined by (2) is right-handed if and only if $\epsilon=1$; thus we have established the invariance of the Cartesian formula for the curl under right-handed orthogonal transformations.

For a general value of $n$ there are, of course, many possible meanings for $\wedge$, and hence for $\nabla\wedge\mathbf{v}$. Skewness of $\wedge$ will ensure that gradients (of scalar fields having continuous second-order derivatives) are irrotational; for,

$$\nabla\wedge\operatorname{grad}\phi=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\phi}{\partial x_i\partial x_j}\,\mathbf{e}_i\wedge\mathbf{e}_j=\mathbf{0}$$

if $\mathbf{e}_i\wedge\mathbf{e}_j=-\mathbf{e}_j\wedge\mathbf{e}_i$, since $\partial^2\phi/\partial x_i\partial x_j=\partial^2\phi/\partial x_j\partial x_i$. Similarly, the divergence of $\nabla\wedge\mathbf{v}$ will be zero if the multiplication table has the property $\mathbf{e}_i\wedge\mathbf{e}_j\cdot\mathbf{e}_k=-\mathbf{e}_k\wedge\mathbf{e}_j\cdot\mathbf{e}_i$.

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## CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

*All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.*

### ON THE CONVERGENCE OF THE SERIES $\sum n^{-\alpha}$

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**1.** Here is a simple proof of the convergence of $\sum n^{-\alpha}$ for $\alpha>1$. The idea originates from a recent note by V. P. Palamodov [1] where he begins with putting

$$1^{-\alpha}+2^{-\alpha}+3^{-\alpha}+\cdots=A,$$

admitting $A$ to be "finite or infinite!—that we don't know yet." He then observes that

$$2^{-\alpha} + 4^{-\alpha} + 6^{-\alpha} + \cdots = 2^{-\alpha}A$$
$$3^{-\alpha} + 5^{-\alpha} + 7^{-\alpha} + \cdots \leqq 2^{-\alpha}A,$$

and therefore

$$A = 1 + (2^{-\alpha} + 4^{-\alpha} + \cdots) + (3^{-\alpha} + 5^{-\alpha} + \cdots)$$
$$\leqq 1 + 2^{-\alpha}A + 2^{-\alpha}A \quad \text{whence} \quad A \leqq 2^{\alpha-1}(2^{\alpha-1} - 1)^{-1}.$$

At the cost of a few more lines this proof is readily given in a more conventional form avoiding inequalities involving the symbol $A$ of which it is to be shown that it represents a positive number, and which *a priori* could also be $\infty$.

Let

$$A_n = 1 + 2^{-\alpha} + 3^{-\alpha} + \cdots + n^{-\alpha} \qquad (n = 2, 3, \cdots).$$

Then for all $n \geqq 2$

$$2^{-\alpha} + 4^{-\alpha} + \cdots + (2n-2)^{-\alpha} + (2n)^{-\alpha} = 2^{-\alpha}A_n$$
$$3^{-\alpha} + 5^{-\alpha} + \cdots + (2n-1)^{-\alpha} \qquad < 2^{-\alpha}A_n.$$

Consequently

$$A_{2n} < 1 + 2^{-\alpha}A_n + 2^{-\alpha}A_n = 1 + 2^{1-\alpha}A_n \qquad (2^{1-\alpha} < 1).$$

In particular for $m = 2, 3, \cdots$

$$A_{2^m} < 1 + 2^{1-\alpha}A_{2^{m-1}} < 1 + 2^{1-\alpha} + (2^{1-\alpha})^2 A_{2^{m-2}} < \cdots$$
$$< 1 + 2^{1-\alpha} + (2^{1-\alpha})^2 + \cdots + (2^{1-\alpha})^m < 2^{\alpha-1}(2^{\alpha-1} - 1)^{-1}.$$

Thus we have a bound, independent of $m$ for all $A_{2^m}$. In the usual way we conclude that this is also a bound for all $A_n$. Hence $\lim_{n\to\infty} A_n$ exists. The bound is the same as that obtained by Palamodov for his sum value $A$.

In this form the proof is, in its idea, similar to the one based on Cauchy's convergence theorem (cf. [2], pp. 118–119).

In a similar way one can decide about divergence of $\sum n^{-\alpha}$ if $0 < \alpha < 1$. Indeed, still

$$2^{-\alpha} + 4^{-\alpha} + \cdots + (2n)^{-\alpha} \qquad = 2^{-\alpha}A_n$$

and therefore

$$1^{-\alpha} + 3^{-\alpha} + \cdots + (2n-1)^{-\alpha} > 2^{-\alpha}A_n.$$

Thus

$$A_{2n} > 2^{1-\alpha}A_n \qquad (2^{1-\alpha} > 1),$$

and in particular

$$A_{2^m} > 2^{1-\alpha}A_{2^{m-1}} > (2^{1-\alpha})^2 A_{2^{m-2}} > \cdots > (2^{1-\alpha})^m,$$

which proves the divergence. The case $\alpha=1$ escapes, whereas it is covered by the test in Cauchy's theorem.

**2.** The same principle can be applied in more general cases. We prove the following

THEOREM. *Let $A=a_1+a_2+\cdots$ be an infinite series with monotonically decreasing positive terms, $p$ a natural number $>1$, and $\gamma$ a positive real number such that $p\gamma<1$. Also let $A_n=a_1+\cdots+a_n$. The series $A$ is convergent if for all $n=1, 2, \cdots$ the sum*

$$(*) \qquad a_p + a_{2p} + \cdots + a_{(n-1)p} + a_{np} \leqq \gamma A_n.$$

*Moreover then $A \leqq A_{p-1}(1-p\gamma)^{-1}$.*

*Proof.* Because by supposition $a_p > a_{p+1} > \cdots$ we conclude from (*) that

$$\begin{aligned} a_{p+1} + a_{2p+1} + \cdots + a_{(n-1)p+1} &< \gamma A_n \\ a_{p+2} + a_{2p+2} + \cdots + a_{(n-1)p+2} &< \gamma A_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{2p-1} + a_{3p-1} + \cdots + a_{np-1} &< \gamma A_n \end{aligned}$$

and therefore

$$A_{np} < A_{p-1} + p\gamma A_n.$$

In particular

$$\begin{aligned} A_{p^m} &= A_{p-1} + p\gamma A_{p^{m-1}} < A_{p-1} + p\gamma A_{p-1} + (p\gamma)^2 A_{p^{m-2}} < \cdots \\ &< A_{p-1}(1 + p\gamma + (p\gamma)^2 + \cdots + (p\gamma)^{m-1}) + (p\gamma)^m A_1. \end{aligned}$$

Thus if $p\gamma<1$ we have for all $m$

$$A_{p^m} < A_{p-1}(1 - p\gamma)^{-1},$$

that is, a bound independent of $m$. This is also an upper bound for all $A_n$.

Conversely if there is for a fixed $p>1$ a positive $\delta$ such that for all $n$

$$a_p + a_{2p} + \cdots + a_{np} > \delta A_n$$

for all positive integers $n$, then it is readily shown that

$$A_{np} > p\delta A_n$$

and we conclude that the series $\sum a_n$ is divergent if $p\delta>1$.

**References**

1. V. P. Palamodov, O ckhodimosti ryada $1+2^{-\alpha}+3^{-\alpha}+\cdots$, Mat. Prosv., vol. 1, 1957, p. 148.

2. K. Knopp, Theorie und Anwendung der unendlichen Reihen, Berlin, 1924.

## AN ABSTRACTION OF A COMBINATORIAL CONCEPT

A. A. MULLIN, University of Illinois

This note is concerned with giving a set-theoretical abstraction of the usual definition of the concept of combinations that appear in combinatorial analysis [1] and to show an anomaly that occurs in the transfinite case when compared to the finite case. We adopt the notation of Cantor [2] in representing cardinal numbers (*i.e.*, the double-bar notation).

DEFINITION. *Let $A$ and $B$ be sets with $B \subseteq A$. Let $\phi$ be a mapping. The cardinal-combinations of $\overline{\overline{A}}$ taken $\overline{\overline{B}}$ at a time, denoted $(\overline{\overline{A}}/\overline{\overline{B}})$, is $\overline{\overline{\Lambda}}$, where $\Lambda = \{\phi : \phi(C) = D$, for some $C$ for which $\overline{\overline{A}} = \overline{\overline{C}}$ and for all $D$ for which $\overline{\overline{B}} = \overline{\overline{D}}\}$.*

From an interpretation of the binomial theorem it is trivial to show that for every $A$ for which $\overline{\overline{A}}$ is finite,

$$(\overline{\overline{A}}/\overline{\overline{B}}) < 2^{\overline{\overline{A}}}, \qquad \text{for all } B \subseteq A.$$

THEOREM. *For every $A$ for which $\overline{\overline{A}} \geqq \aleph_0$, $\exists B \subseteq A$ such that $(\overline{\overline{A}}/\overline{\overline{B}}) = 2^{\overline{\overline{A}}}$.*

*Proof.* Put $\alpha = \{C : C \subseteq A\}$ and put $\beta = \sum_{B \subseteq A} (\overline{\overline{A}}/\overline{\overline{B}})$.† It can be shown [3] that $\overline{\overline{\alpha}} = 2^{\overline{\overline{A}}}$. But by the algebra of cardinal numbers [4] and our definition, $\overline{\overline{\alpha}} = \sum_{B \subseteq A} (\overline{\overline{A}}/\overline{\overline{B}})$. No term of $\beta$ can be greater than $2^{\overline{\overline{A}}}$, for otherwise, by the algebra of cardinal numbers $\sum_{B \subseteq A} (\overline{\overline{A}}/\overline{\overline{B}}) > 2^{\overline{\overline{A}}}$. Not all terms of $\beta$ can be strictly less than $2^{\overline{\overline{A}}}$, for otherwise, by the generalized continuum hypothesis and the algebra of cardinal numbers $\sum_{B \subseteq A} (\overline{\overline{A}}/\overline{\overline{B}}) \leqq \overline{\overline{A}} < 2^{\overline{\overline{A}}}$.

It is well known [5] that there exists a relation $R$ for which every collection of cardinal numbers can be well-ordered with respect to $R$ and, *a fortiori*, every pair of cardinal numbers is comparable. (In this respect, Kelley's last sentence in the first paragraph on page 28 seems to have a contradiction in it, as per his theorem 162 of the appendix, although this example is probably the " . . . single nontrivial exception . . . " that he refers to in the footnote on his page 1.)

Hence by comparability and the law of excluded-middle the existence of the required $B$ is established.

### References

1. J. Riordan, An Introduction to Combinatorial Analysis, New York, 1958.
2. G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, New York, 1915.
3. E. Kamke, Theory of Sets, New York, 1950.
4. F. Hausdorff, Set Theory, New York, 1957.
5. J. L. Kelley, General Topology, New York, 1955.

† At all its appearances this summation is for distinct $\overline{\overline{B}}$'s.

## SEQUENCES GENERATED BY USE OF THE MEAN VALUE THEOREM

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Classroom presentations of applications of the mean value theorem often include obtaining first approximations to given functional values $f(a)$. It is interesting to note that under certain conditions the mean value theorem may yield a good deal more. The theorem can be used to generate a sequence $\{f(b_n)\}$ which actually converges to $f(a)$. The result is stated in the following theorem.

THEOREM. *Let $f'$ be continuous and decreasing in the closed interval $[a, b]$ and assume that $f'(b)>0$. Then the numbers $b_n=f^{-1}[f(b_{n-1})-f'(b_{n-1})(b_{n-1}-a)]$ exist, with $a<b_n<b_{n-1}$, for $n=1, 2, \cdots$, and with $b_0=b$. The sequence $\{f(b_n)\}$ decreases to $f(a)$.*

The first part of the theorem is established by induction. By the mean value theorem applied to $f$ in $[a, b]$, there exists an $X$, $a<X<b$, such that $f(a)=f(b)-f'(X)(b-a)$. Then, since $f'$ is decreasing and $f'(b)$ is positive, $f(a)<f(b)-f'(b)(b-a)<f(b)$. It follows from the hypothesis of the theorem that $f$ is increasing and continuous in $[a, b]$. Hence, $f^{-1}$ exists, is increasing and continuous in $[f(a), f(b)]$, and thus $b_1=f^{-1}[f(b)-f'(b)(b-a)]$ exists, with $a<b_1<b$. Now assume $b_n$ exists with $a<b_n<b_{n-1}$. As before, application of the mean value theorem to $f$ in $[a, b_n]$ yields $f(a)<f(b_n)-f'(b_n)(b_n-a)<f(b_n)$, so the number $b_{n+1}=f^{-1}[f(b_n)-f'(b_n)(b_n-a)]$ exists with $a<b_{n+1}<b_n$ and our induction is complete.

The sequence $\{b_n\}$ is decreasing and bounded below by $a$, so $\lim_{n\to\infty} b_n$ exists. Denoting this limit by $x_0$, we have $a \leqq x_0 < b$. Since $f(b_n) = f(b_{n-1})-f'(b_{n-1})(b_{n-1}-a)$, it follows from the continuity of $f$ and $f'$ at $x_0$ that $f(x_0)=f(x_0)-f'(x_0)(x_0-a)$. Thus, $x_0=a$ and $\lim_{n\to\infty} f(b_n)=f(a)$. The sequence $\{f(b_n)\}$ is decreasing because $\{b_n\}$ is.

The problem of approximating roots affords a simple application of the theorem. For example, to approximate $\sqrt{3}$ we take $f(x)=\sqrt{x}$, $a=3$, $b=4$. Since the values of $f^{-1}$ are easily determined and $f'(b_n)$ can be expressed in terms of $f(b_n)$, the successive values of $f(b_n)=\sqrt{b_n}$ are readily found: $\sqrt{b_1}=\sqrt{b}-(b-a)/2\sqrt{b}=7/4=1.75$, $\sqrt{b_2}=\sqrt{b_1}-(b_1-a)/2\sqrt{b_1}=97/56=1.73214\cdots$ and so forth.

If $f'$ is continuous and increasing in $[a, b]$ with $f'(a)>0$, the mean value theorem can be used as above to generate a sequence $\{f(a_n)\}$ increasing to $f(b)$; here $a_n=f^{-1}[f(a_{n-1})+f'(a_{n-1})(b-a_{n-1})]$. In the case where $f$ is decreasing in $[a, b]$ we need merely replace $f(x)$ by $-f(x)$.

* The author wishes to express her appreciation of suggestions made by the referee.

## IRREDUCIBILITY OF POLYNOMIALS

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Van der Waerden [1] states and proves a criterion for reducibility of polynomials which is attributed to G. Dumas [2]. It is the purpose of this note to call attention to the stronger and more useful formulation of this result as stated and proved in [3]. It will be assumed that $f(x)$, $g(x)$ and $h(x)$ are polynomials in $x$ with integral coefficients, $p$ is a prime, and $a_i$ is an integer relatively prime to $p$. Let

$$f(x) = \sum_{i=0}^{n} a_i p^{b_i} x^i$$

and let the points $(i, b_i)$ be plotted in the usual way. From this set of points a subset $\{P_j\}_{j=0}^{r}$ is generated by choosing $P_0$ as the point $(0, b_0)$ and $P_j$ $(j=1, \cdots, r)$ as $(k_j, b_{k_j})$, where $k_j$ is the greatest integer such that no point $(i, b_i)$ lies below the line through $P_{j-1}$ and $P_j$. $P_r$, of course, is $(n, b_n)$. The figure composed of the line segments $P_{j-1}P_j$ for $j=1, \cdots, r$ is called the Newton polygon for $f(x)$ corresponding to the prime $p$. If for example $f(x)=27+6x^3+4x^4+6x^5+12x^6$, the Newton polygons corresponding to 3 and 2 appear in Figure 1.

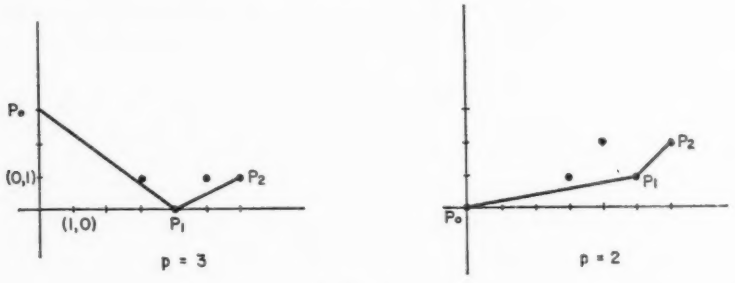


FIG. 1

THEOREM OF DUMAS. *Let the segments of the Newton polygon for $f(x)$ corresponding to $p$, be subdivided by the lattice points occurring on them and let the resulting segments connecting adjacent points of division be called the elements of the polygon. If $f(x)=g(x)h(x)$, then the Newton polygon for $g(x)$ corresponding to $p$ can be formed by joining some of the elements of the polygon for $f(x)$ without changing their lengths or slopes. Moreover, the Newton polygon for $h(x)$ corresponding to $p$ can be formed in the same manner by precisely those elements not used for the polygon of $g(x)$.*

With this theorem it is easy to show that the Newton polygons for the factors of $27+6x^3+4x^4+6x^5+12x^6$ must appear as in Figure 2.

From the polygons for $p=3$ it is necessary that one factor be of degree 2 and the other of degree 4, but for $p=2$ it is indicated that one degree must be 5 and

the other 1. Consequently, $27+6x^3+4x^4+6x^5+12x^6$ is irreducible over the field of rational numbers, whereas van der Waerden's statement of the theorem does not rule out a factorization with polynomials of degrees 5 and 1.

Eisenstein's criterion is an immediate corollary of this theorem and, to the best of the author's knowledge, so also are all of the criteria concerning divisibility of the coefficients.

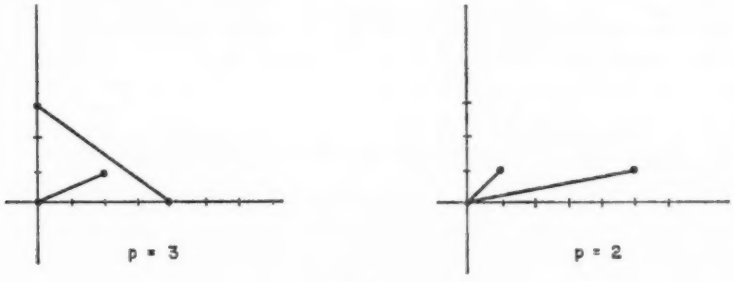


FIG. 2

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1. B. L. van der Waerden, Modern Algebra, New York, 1943, p. 76.

2. G. Dumas, Sur quelques cas d'irréductibilité des polynomes à coefficients rationnels, J. Math. Pures Appl., vol. 2, 1906, pp. 191–258.

3. J. H. Wahab, New cases of irreducibility for Legendre polynomials, Duke Math. J., vol. 19, 1952, pp. 167–169.

## A MATRIX APPLICATION OF NEWTON'S IDENTITIES

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It is well known that the characteristic polynomial of a square matrix can be obtained by computing the traces of the powers of the matrix. Conversely, the trace of any power of a matrix can be obtained from the coefficients of the characteristic polynomial of the matrix. Although these computational techniques follow as an immediate application of Newton's identities on symmetric polynomials, the author is not aware of any mention in the matrix algebra literature of a simple proposition that combines these results. The purpose of this note is to formulate such a theorem for use in the classroom.

We begin with a review of some elementary facts about symmetric polynomials. First, it is well known that any symmetric polynomial in the elements $\alpha_1, \cdots, \alpha_n$ is expressible as a polynomial in the elementary symmetric polynomials $p_1, \cdots, p_n$ in $\alpha_1, \cdots, \alpha_n$. In particular, this applies to the sums of like powers: $s_k=\alpha_1^k+\cdots+\alpha_n^k$. A specific algorithm for successively determining $s_1, s_2, \cdots$ in terms of $p_1, \cdots, p_n$ is provided by Newton's identities:

$$s_1 - p_1 = 0, \quad s_2 - p_1 s_1 + 2p_2 = 0, \cdots,$$
$$s_k - p_1 s_{k-1} + \cdots + (-1)^{k-1} p_{k-1} s_1 + (-1)^k k p_k = 0,$$

where $p_k = 0$ for $k > n$. Furthermore, it is clear that these identities also provide a means of determining $p_1, 2p_2, \cdots, n!p_n$ in terms of $s_1, \cdots, s_n$.

As a consequence of the preceding paragraph, we have the following:

LEMMA. *Let $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_m$ be elements of a field of characteristic zero or prime $p > n \geqq m$. Let $p_k$ and $s_k$ ($q_k$ and $t_k$) denote, respectively, the elementary symmetric polynomials and the sums of like powers of $\alpha_1, \cdots, \alpha_n$ ($\beta_1, \cdots, \beta_m$). Then $p_k = q_k$, $k = 1, \cdots, m$ and $p_k = 0$, $k = m+1, \cdots, n$ if and only if $s_k = t_k$, $k = 1, \cdots, n$.*

As a direct application of this lemma, we now have the following:

THEOREM. *Let $A$ be an $n$-by-$n$ matrix and $B$ an $m$-by-$m$ matrix over a field of characteristic zero or prime $p > n \geqq m$. Then the characteristic polynomial of $A$ is $x^{n-m}$ times the characteristic polynomial of $B$ if and only if* $\operatorname{tr} A^k = \operatorname{tr} B^k$ for $k = 1, \cdots, n$.

*Proof.* Using the notation of the preceding lemma, let

$$(x - \alpha_1) \cdots (x - \alpha_n) = x^n - p_1 x^{n-1} + \cdots + (-1)^n p_n,$$
$$(x - \beta_1) \cdots (x - \beta_m) = x^m - q_1 x^{m-1} + \cdots + (-1)^m q_m,$$

be the characteristic polynomials of $A$ and $B$, respectively, over some extension of the given field. Since the trace of any matrix is equal to the sum of the characteristic values of the matrix, $\operatorname{tr} A^k = \alpha_1^k + \cdots + \alpha_n^k = s_k$ and $\operatorname{tr} B^k = \beta_1^k + \cdots + \beta_m^k = t_k$. The theorem now follows immediately from the lemma above.

It is of interest to note that for the proof of sufficiency in the theorem the restriction on the characteristic of the field is indeed required. This proposition is demonstrated by taking $A$ to be the direct sum of the $p$-by-$p$ identity matrix and the $(n-p)$-by-$(n-p)$ zero matrix and $B$ to be the $m$-by-$m$ zero matrix over a field of characteristic prime $p \leqq n$.

Finally, two corollaries are given. The first gives the known trace criterion for nilpotence of a matrix. (See, for example, [1], p. 10.) The second is very well known over fields of any characteristic, but the simple proof given here is valid only under the restriction of the theorem above. (Compare for example [2], p. 106.)

COROLLARY 1. *Let $A$ be an $n$-by-$n$ matrix over a field of characteristic zero or prime $p > n$. Then $A$ is nilpotent if and only if* $\operatorname{tr} A^k = 0$, $k = 1, \cdots, n$.

*Proof.* This result follows directly from the theorem by taking $B = 0$ and using the fact that $A$ is nilpotent if and only if its characteristic polynomial is $x^n$.

COROLLARY 2. *Let $A$ be an $n$-by-$m$ matrix and $B$ an $m$-by-$n$ matrix over a field*

*of characteristic zero or prime $p > n \geqq m$. Then the characteristic polynomial of $AB$ is $x^{n-m}$ times the characteristic polynomial of $BA$.*

*Proof.* Since the trace of a product of two matrices is the trace of the product in reverse order, $\operatorname{tr} AB = \operatorname{tr} BA$ and $\operatorname{tr}(AB)^k = \operatorname{tr}((AB)^{k-1}A)B = \operatorname{tr} B((AB)^{k-1}A) = \operatorname{tr}(BA)^k$ for $k > 1$. The conclusion is now a consequence of the theorem above.

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1. H. Flanders, Methods of proof in linear algebra, this MONTHLY, vol. 63, 1956, pp. 1–15.
2. N. Jacobson, Lectures in Abstract Algebra, Volume II—Linear Algebra, New York, 1953.

## A DERIVATION OF THE GENERAL SOLUTION FOR HOMOGENEOUS, LINEAR, DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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In obtaining the general solution for homogeneous, linear, difference equations, with constant coefficients, one discusses two cases for the characteristic equation, *i.e.*, (*i*) distinct roots and (*ii*) repeated roots, real or complex. The proofs of (*i*) and (*ii*) use techniques analogous to those used in obtaining the general solution for homogeneous, linear, differential equations with constant coefficients. A typical proof of (*ii*) usually considers only a double root, while roots with higher multiplicities are not treated effectively, since the general result is often stated without proof. In this note, we will establish a theorem which treats cases (*i*) and (*ii*) *simultaneously*, and which reveals a relationship, not generally known, between the solutions of linear difference and linear differential equations.

THEOREM. *Let*

$$(1) \qquad y(x) = \sum_{i=1}^{m} \left\{ \sum_{j=0}^{n_i-1} C_{i(j+1)} x^j \right\} e^{r_i x} + \sum_{j=(n_1+\cdots+n_m)+1}^{n} C_j e^{r_j x}$$

*be the general solution of the following homogeneous, linear, differential equation of nth order:*

$$(2) \qquad D^n y(x) + a_1 D^{n-1} y(x) + a_2 D^{n-2} y(x) + \cdots + a_n y(x) = 0, \qquad a_n \neq 0,$$

*where $C_{i(j+1)}$ and $C_j$ are arbitrary constants, $a_i$, $i = 1, \cdots, n$, are real constants, $n_i \geqq 1$, $i = 1, \cdots, m$, with $(n_1 + \cdots + n_m) \leqq n$, and where the characteristic equation, $r^n + a_1 r^{n-1} + \cdots + a_n = 0$, has the roots, $r_i$, with multiplicity, $n_i$, $i = 1, \cdots, m$, and the simple roots, $r_j$.*

*Let $u_k$ be the general solution of the following homogeneous, linear, difference equation of nth order:*

$$(3) \qquad u_{n+k} + a_1 u_{n+k-1} + a_2 u_{n+k-2} + \cdots + a_n u_k = 0.$$

*Then $u_k = D^k y(0)$ and*

$$u_k = \sum_{i=1}^{m} \left\{ C_{i1} + \sum_{\nu=1}^{n_i-1} \gamma_{i\nu} k^{\nu} \right\} r_i^k + \sum_{j=(n_1+\cdots+n_m)+1}^{n} C_j r_j^k, \tag{4}$$

*where the $\gamma_{i\nu}$ are arbitrary constants.*

*Proof.* If we differentiate (2) $k$ times with respect to $x$ and then set $x=0$, we obtain (3), where $u_k = D^k y(0)$. Let $A_k(x) = D^k\left[\sum_{j=0}^{n_i-1} C_{i(j+1)} x^j e^{r_i x}\right]$. Using Leibnitz's rule, we obtain

$$\begin{aligned} A_k(x) &= C_{i1} r_i^k e^{r_i x} + \sum_{j=1}^{n_i-1} C_{i(j+1)} \left\{ \sum_{p=0}^{k} \binom{k}{p} r_i^{k-p} e^{r_i x} D^p x^j \right\} \\ &= C_{i1} r_i^k e^{r_i x} + \sum_{j=1}^{n_i-1} C_{i(j+1)} \sum_{p=0}^{j} \binom{k}{p} r_i^{k-p} e^{r_i x} j^{(p)} x^{j-p}, \end{aligned} \tag{5}$$

where $k^{(j)} = k(k-1)\cdots(k-j+1) = \sum_{\nu=1}^{j} S_\nu^j k^\nu$, ([1], p. 142), and $S_\nu^j$ is a Stirling number of the first kind. Setting $x=0$ in (5), we obtain

$$\begin{aligned} A_k(0) &= C_{i1} r_i^k + \sum_{j=1}^{n_i-1} C_{i(j+1)} \binom{k}{j} r_i^{k-j} (j!) = C_{i1} r_i^k + \sum_{j=1}^{n_i-1} C_{i(j+1)} r_i^{k-j} k^{(j)} \\ &= C_{i1} r_i^k + \sum_{j=1}^{n_i-1} C_{i(j+1)} r_i^{k-j} \sum_{\nu=1}^{j} S_\nu^j k^\nu = C_{i1} r_i^k + \sum_{\nu=1}^{n_i-1} \left( \sum_{j=\nu}^{n_i-1} C_{i(j+1)} r_i^{k-j} S_\nu^j \right) k^\nu. \end{aligned}$$

For each $i$, $i=1, \cdots, m$, set $C_{i(j+1)} = \beta_{ij} r_i^j$, $j=1, \cdots, n_i-1$, and $\gamma_{i\nu} = \sum_{j=\nu}^{n_i-1} \beta_{ij} S_\nu^j$, $\nu=1, \cdots, n_i-1$. Thus

$$A_k(0) = C_{i1} r_i^k + \sum_{\nu=1}^{n_i-1} \left( \sum_{j=\nu}^{n_i-1} \beta_{ij} S_\nu^j \right) r_i^k k^\nu = \left[ C_{i1} + \sum_{\nu=1}^{n_i-1} \gamma_{i\nu} k^\nu \right] r_i^k.$$

*Remarks.* The example, $y''(x) = y'(x) + y(x)$, is interesting, since the associated difference equation, $u_{k+2} = u_{k+1} + u_k$, generates a sequence of Fibonacci numbers. The choice of $x=0$ in the theorem is convenient, since it minimizes the number of notational changes for the arbitrary constants.

If the solution of (2), given by (1), is written as a power series, $y(x) = \sum_{k=0}^{\infty} \sigma_k x^k$, where $\sigma_k = D^k y(0)/k!$, then $u_k = k!\sigma_k$. Thus (3) yields, after substituting for $u_k$ and then dividing through by $k!$, the recursion formula satisfied by the series coefficients, $\sigma_k$.

It should be noted that the theorem can be extended to nonhomogeneous, linear, difference equations of $n$th order. For example, if the right-hand side of (3) is given by $\sum_{i=1}^{N} \mu_i (\eta_i)^k$, where $\mu_i$ and $\eta_i$ are specified, real constants, and if the right-hand side of (2) is given by $\sum_{i=1}^{N} \mu_i e^{\eta_i x}$, then the formula $u_k = D^k y(0)$, is still valid, where $y(x)$, given by (1), now includes the particular integral for (2), as modified.

**Reference**

1. Charles Jordan, Calculus of Finite Differences, New York, 1950.

### SPOOF OF THE FUNDAMENTAL THEOREM OF CALCULUS

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A "spoof" is hereby defined as a short proof with much missing. This is not inconsistent with Webster's definition as "hoax."

The integral undoes differentiation since

$$\text{Derivative} = \text{Limit of } [\text{Quotient of Differences}]$$

$$\text{ANTIderivative} = \text{Limit of } [\text{Sum of Products}]$$

## MATHEMATICAL EDUCATION NOTES

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### SUPPORT OF HIGHER EDUCATION BY THE FEDERAL GOVERNMENT*

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In discussing support of higher education by the Federal Government we should recognize that although there is still much discussion of the pros and cons of such support, and although there has been no overall policy decision by the Congress or by the executive branch of the government or by any group of representative leaders of American higher education that such support should be given, it is true nonetheless that a substantial amount of support is currently provided to institutions of higher education by various agencies of the national government. This support is estimated as between 1.5 and 2.0 billion dollars per year. To quote from an article by Homer D. Babbidge, Jr., Assistant Commissioner and Director, Division of Higher Education, Office of Education:

> "A half-dozen different graduate fellowship programs are now in operation, each administered by a different Federal agency. Some 6,000 students are being financed through these programs in their graduate work, at an annual Federal cost of $35 million. Approximately 75 percent of these awards are in mathematics and the sciences, and 17 percent are in the humanities and social sciences, which account for slightly more than a fourth of the Ph.D.'s. The average Federal fellowship in the sciences pays the student $700 more per year than the fellowship in a nonscience field.
>
> "Five Federal agencies conduct major research programs, using U.S. universities as the principal resource. This year they will spend at least $750 million for such research, $450 million directly in universities and the remaining $300 million in research centers associated

* Keynote speech at a meeting on Federal Support of Higher Education called by the Conference Board of the Mathematical Sciences, November 12–13, 1960, Washington, D. C.

with universities. As a result, more than 70 percent of all research conducted by our universities is federally financed: 86 percent of all university research in the physical sciences and 25 percent in the social sciences. C. V. Kidd, whose studies of this area of Federal activity are highly illuminating, reports that 14 universities received 55 percent of Federal research funds in 1953–54 and that 10 universities received about one-third in 1959.

"A recent comparison of 10 large universities with 10 small colleges indicates that the universities received about 33 percent of their general educational income from Federal sources, while the small institutions received about 3 percent of their income from the Federal Government.

"Legislation pending before the Congress would extend Federal participation in higher education to include physical plant construction. Leaders of both political parties and both the executive and legislative branches of Government are agreed to such an extension, and disagree only on methods and amounts. During the last 3 years, $1 out of every $4 spent on higher education construction throughout the nation came from Federal sources—the college housing loan program.

"More than 135,000 young men and women are receiving Federal financial assistance amounting to more than $75 million this year under the National Defense Student Loan Program. Borrowing for the financing of colleges has become an established fact as a direct result of this legislation.*

"The teacher-training function of the nation's colleges and universities has been considerably expanded through the conduct of federally financed institutes for secondary school personnel in mathematics, science, modern foreign languages, and counseling and guidance. No Federal funds have been made available for such institutes in the humanities or social sciences."

In this article I shall discuss the nature and needs for Federal support as these relate to mathematics. It may be worthwhile to consider briefly certain aspects of the history of these questions. At the end of World War II the urgent need of the Government for new scientific results, sometimes in narrowly delimited fields, and the recognition that the universities of the United States were not planning and were financially unable to embark on a program of basic research in the sciences adequate to provide for these needs, led to the decision to initiate a widespread program of Federal support of university research in the sciences.

The first steps in this new program (and we should not forget already existing efforts in the universities like those of the Department of Agriculture) were taken by the Navy through the Office of Naval Research. With the passage of time many other agencies entered the field. The establishment of the National Science Foundation in 1950 gave an added impetus to this development.

Mathematics was early recognized as an essential element in any vigorous program for the development of leadership in the sciences, and this recognition included the purest of mathematical disciplines as well as applied mathematics and the development of computers. As more extensive programs of support have emerged the position of mathematics has been maintained, not however without the determined and selfless effort of many leaders in the profession. It is worth noting that the nature of the support needed for mathematicians is different from that needed for experimental scientists.

* Higher Education, vol. 17, 1960, p. 4.

There were two main reasons for federal contributions toward the support of research in the experimental sciences: (1) the need for expensive equipment which in many fields like nuclear physics grew more and more expensive with the passage of time; and (2) the need for results in particular areas of research and for basic advances across a broad front of scientific inquiry. The first of these reasons was not present in mathematics except in computer development, in which mathematicians participated with engineers, physicists and other scientists.

Though the criteria for the support of research programs that have been emphasized in the past are primarily the two I have listed, a third criterion, the need for trained personnel, was constantly in the picture. As we assess our situation in mathematics today, this need for trained personnel becomes the critical issue. To the extent that this personnel need is acute in other fields, like the social sciences and certain of the humanities, it may be that programs comparable to those suggested for mathematics should be considered.

In recent years the nature of the involvement of higher education in federal programs has changed. Federally supported research programs continue, but the extent to which research people in the arts and sciences are now committed to participation in course content improvement programs at the secondary and even at the elementary level has made new demands on the time of scholars. And in mathematics the Mathematical Association, through the work of the Committee on the Undergraduate Program, has emerged as a significant force in the modernization of collegiate mathematics courses as well as in the design of radical new plans for the training of teachers to handle all the new curricula. All this takes people. Federal government money for research programs tends to go to a relatively few of the large universities with the effect of strengthening these institutions and increasing their power to attract the ablest scholars. Too often they are attracted away from the smaller liberal arts colleges. Moreover, the teachers at the liberal arts colleges often lack the stimulation and vitality gained from actual participation in research. The National Science Foundation, through its education projects, has come to grips with the problem of providing better subject matter competence to teachers in liberal arts colleges and in secondary schools. One need only mention some of the programs of the Foundation—

> Science Faculty Fellowships, Summer Fellowships for Secondary School Teachers, Summer Institutes for High School Teachers of Science and Mathematics, Academic Year Institutes for High School Teachers of Science and Mathematics, In-Service Institutes for High School Teachers of Science and Mathematics, Summer Conferences for College Teachers, Visiting Scientists Programs for Colleges and for Secondary Schools

to recognize the thoroughness with which the Foundation has identified the needs, and the imagination with which it has embarked on programs to try to meet those needs.

For mathematics the outstanding need is for more trained mathematicians.

If we give our attention for a moment to the need for teachers in universities, colleges, and two-year colleges, we see how critical is our situation with regard to the production of Ph.D.'s. In 1953–54 34.2% of all new mathematics teachers in 4-year colleges had Ph.D.'s; in 1958–59 only 19.9% of such teachers had Ph.D.'s.* In 1930 about 15% of the Ph.D.'s in mathematics were in non-academic employment.† This percentage has been rising steadily, except immediately after World War II. Preliminary results indicate that in 1960, for the first time, a little over half of the approximately 300 new mathematics Ph.D.'s entered industry or government (final results are not available at this writing).

What is the prospect for providing the teachers to handle the crowds of new students? Except during the war the number of mathematics Ph.D.'s has constituted an almost constant percentage of all science and mathematics Ph.D.'s (a little under 5%) and also of all Ph.D.'s in all fields (around 2.8%).‡ Thus if the total number of Ph.D.'s in all fields does double within the next decade along with the doubling of the college population, as has been predicted, we may hope to produce around 600 Ph.D.'s in mathematics in 1970. But a little arithmetic will show a terrifying imbalance even if, as seems most unlikely, the drift to industry and government were stemmed.

The junior year enrollments of mathematics majors increased by more than 30% between the fall of 1957 and the fall of 1958, while the increase was only a little over 6% for the biological sciences and about 4% for the physical sciences. During this period total junior year enrollments in institutions of higher education increased by less than 4%.§ Thus enrollment of mathematics majors increased about eight times as fast as the total enrollment. If the enrollments in mathematics continue to climb at this rate while the production of Ph.D.'s merely keeps pace with the total enrollment, the picture seems black. The situation calls for the most imaginative combined efforts of the mathematical community if we are to produce more people qualified to carry on the work that relies on mathematicians: more research in mathematics, pure and applied; more use in applications, not only in the traditional fields like physics and engineering but also in new fields like economics, and in even newer fields like biology and medicine (and here I refer not only to the applications of statistics, but to the efforts to evolve mathematical models to provide insights that are only now suggesting themselves, into the unsolved problems of biological research). More teachers are needed for the increasing number of students preparing to enter these varied fields. This need for teachers is the most acute need of all.

Why is it that the yield of mathematical doctorates is so small a proportion of all doctorates in science and mathematics together? Why is it that so few

* Statistical Handbook of Science Education, National Science Foundation, NSF 60-13, p. 82.

† A Survey of Research Potential and Training in the Mathematical Sciences, Part 1, The University of Chicago, March 15, 1957, p. 109.

‡ Statistical Handbook of Science Education, National Science Foundation, NSF 60-13, p. 79.

§ Statistical Handbook of Science Education, National Science Foundation, NSF 60-13, p. 69.

mathematics majors go on to the Ph.D.? Must a student be a genius to receive a Ph.D. in mathematics? Some of our students seriously think the answer to this question is "Yes." In physics, a B student at college can do a very good job in his Ph.D. research; but a B student in mathematics will rarely be accepted as a candidate for a doctorate in mathematics. We shall certainly need some of our B students as teachers, particularly in our two-year colleges if these continue to spring into being as they have been doing recently. In several states, the master plan for the development of higher educational facilities to take care of the doubling of collegiate enrollment within the next decade calls for the establishment of many new community and junior colleges within a few years.*

How shall we staff their mathematics faculties? What can the mathematical community do to attract more able young people into graduate work in mathematics? Are the graduate departments of mathematics getting their share of the mathematically gifted students?

In this connection there is one program of the National Science Foundation which seems to me threatening. This is, in itself, an excellent undertaking, but it was designed without the inclusion of an adequate provision for mathematics. It provides financial assistance to undergraduates who participate with faculty members in scientific research programs in liberal arts colleges. Because of the nature of mathematical research, there has been only a handful of proposals for support in genuinely mathematical projects. These were submitted to the Foundation with a mass of proposals in the experimental sciences. The result has been that many liberal arts colleges were given funds with which to pay honor students in the experimental sciences for their undergraduate research while no comparable awards are available for mathematics honor students. The possibility that this lure may prove sufficient to drain off some of the few promising prospective mathematicians into experimental sciences is worrisome. It seems important that mathematicians develop programs that the Foundation will be willing to consider as comparable to the experimental science research programs so that mathematics undergraduates will also receive financial assistance and have special opportunities, while they are still undergraduates, to taste the flavor of creative work in mathematics. The importance of the financial motivation should not be underestimated, particularly for some of our gifted but financially pressed young people. Paid summer work may, in some cases, determine the choice of a career.

How are we to take account of the vastly expanded need for new mathematicians, particularly new teachers of mathematics? The situation is so desperate that very serious consideration should be given to the proposal from the Committee on the Undergraduate Program to the Mathematical Association of America and the American Mathematical Society that a new doctoral degree be created. We should note that there are now some schools of education that

* See, for example, Meeting the Increasing Demand for Higher Education in New York State, A Report to the Governor and the Board of Regents, November, 1960, p. 30.

award the Ph.D. degree in mathematics on the basis of course work devoted half to mathematics and half to education courses, and followed by a thesis that is expository or historical. If we keep this fact in mind, it may not be quite so shocking to contemplate a doctor's degree based on course work probably more extensive and less intensive than the course work now used as the basis for a Ph.D. (but given at the same level as is now customary at the best graduate schools) followed by a thesis that is critical or expository. Such a degree could be expected to provide some of the teachers for liberal arts colleges and many of the mathematicians who enter industry. These men would be equipped with the kind of background they most need. Such a degree could be the basis for a steady push to prevent the widespread staffing of our institutions of higher education with faculties whose education in mathematics is meager. If such preparation were combined with constant efforts for continuing education like that provided in the National Science Foundation Summer Conference Program, we might look forward to liberal arts mathematics faculties that are mathematically mature and alert, in spite of the frightening pressure that is developing. Even in 1953–54, only about one-fourth of mathematics faculty members spent any time on research; so that the programs of the National Science Foundation designed to keep liberal arts faculty members in touch with new results seem to be our best hope for maintaining and improving the quality of mathematical education. The introduction of the proposed degree would have no significant effect on the Ph.D. awarded for original research. The real test of a creative mathematician is not, of course, his possession of a degree, but his contributions toward the advancement of mathematics; and there seems to be nothing in the proposed program that would threaten the development of genuine mathematical talent. Moreover, the trend toward a postdoctoral year or more for the really creative mathematician is so clearly established that it seems natural to look to this kind of experience to develop the gifted mathematical leaders of the next generation.

Our first and outstanding requirement seems to be that we make whatever adjustments are possible in our educational system to encourage the production of more persons equipped to perform the mathematical tasks that must be performed in the years immediately ahead. Where will we look for the additional people to perform these tasks? One immediate answer here, as it has been in many comparable situations, must be, "What of women?" We know, historically, that women have characteristically not been productive mathematicians. The few efforts I know of that have attempted to determine reasons for this fact have not produced results of great importance. But whether or not we understand the reason, there seems to be no question of the fact. In teaching, however, and in many phases of applied mathematics that are now claiming large numbers of our trained youth, women have performed most effectively. It would seem likely, that, with a degree such as that suggested by the Committee on the Undergraduate Program, and with serious efforts to persuade guidance counselors in the nation's high schools that able and interested girls

should be encouraged rather than discouraged from thinking of mathematics as a career, and, of course, with a serious effort to acquaint girls with the increasing trend of women back to the labor market after they have raised their children, we might hope to draw additional numbers of young women into careers in mathematics. Though my feminine loyalties would tend to urge that extensive and impressive improvement in the quality of the whole mathematical effort would occur if we could achieve this end, my honest assessment of the situation leads me to the conclusion that we would benefit primarily on the quantitative side. However, the effort seems to me critically important.

I have spoken only of the need for people with the mathematical sophistication and dedication necessary to carry on the emerging work of our world. Though this is certainly the outstanding demand that the times make upon us, there are clearly other programs that might move us forward in achieving better uses of the men we have. These I shall not attempt to delineate here for I want to emphasize my belief that our biggest problem is to produce more mathematicians at *all* levels. We should not lose a single qualified student because of race, sex, or the multitudes of accidents that determine the choice of an undergraduate college. We need all the imagination we can bring to bear to secure more nearly our share of able youth to carry on the research, the teaching, and the applications functions that this generation of mathematicians must handle. We must use every device in our power to insure sound and stimulating undergraduate opportunities for all the mathematically gifted youth in the land, and to attract these young people into the careers in mathematics for which their talents and interests equip them, and for which the needs of our society make so urgent an appeal.

### The Advanced Studies Program at St. Paul's School

The Advanced Studies Program of St. Paul's School, Concord, New Hampshire, will for the first time this coming summer admit girls to its six weeks session. With the approval of principals, superintendents, and the State Department of Education, the Program supplements and enlarges upon New Hampshire public high school courses. For each of the past three summers the Program has enrolled more than one hundred academically talented students from New Hampshire public and parochial high schools. The Reverend Matthew M. Warren, Rector of St. Paul's School, in making the announcement of this change, pointed out that the Program has always been anxious to include girls as well as boys. He said that the limitations of financial aid to the Program and the fact that students have been able to pay an average of only half the cost, have prevented the Program from increasing its contribution to education in the State of New Hampshire.

### Noble Memorial Lecture on the Teaching of Mathematics

The first Miss Charlie M. Noble Memorial Lecture on the Teaching of Mathematics was given, December 3, 1960, by Dr. H. J. Ettlinger, Professor of Mathematics at the University of Texas, at Texas Christian University. The lecture was sponsored by the Fort Worth Public Schools, Texas Christian University and the Texas Academy of Science. Miss Noble taught mathematics sixty-two years, most of them in Central High School and R. L. Paschal Senior High School of Fort Worth and at Texas Christian

University. She also for many years taught junior astronomy groups at the Children's Science Museum of Fort Worth. Professor Ettlinger has taught mathematics at the University of Texas for forty-eight years.

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## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 1461. *Proposed by Underwood Dudley, University of Michigan*

Solve the cryptic addition

$$\begin{array}{r} \text{FIFTY} \\ \text{FOUR} \\ \text{FOUR} \\ \text{TWO} \\ \hline \text{SIXTY} \end{array}$$

remembering that FOUR$+12$ is a perfect square.

E 1462. *Proposed by Michael Skalsky, Southern Illinois University*

In how many ways can 4 white, 3 black, and 3 red balls be arranged in a row so that no two adjacent balls are of the same color?

E 1463. *Proposed by V. F. Ivanoff, San Carlos, California*

Given an imaginary point $P$: $(a+pi, b+qi, c+ri)$ in 3-space. Find the locus of real points $(x, y, z)$ whose distance from $P$ is real.

E 1464. *Proposed by Freddy Storey, Princeton University*

Show that for $n \geqq 2$, $\prod_{i=0}^{n} \binom{n}{i} \leqq \{(2^n-2)/(n-1)\}^{n-1}$.

E 1465. *Proposed by U. R. Kodres, IBM Corp., Poughkeepsie, New York*

It is well known that the graph which represents the classical problem of connecting three houses to three utilities is not planar. Prove that a generalization of this graph, namely the graph which represents connecting seven houses to seven utilities is not biplanar, *i.e.*, the graph cannot be factored into two planar factors.

## SOLUTIONS

### A Non-Linear Difference Equation

E 1431 [1960, 802]. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Advanced Development*

If $a_{n+1}=(1+a_na_{n-1})/a_{n-2}$ and $a_1=a_2=a_3=1$, show that $a_n$ is an integer.

I. *Solution by J. L. Pietenpol, Columbia University.* Define a sequence $\{b_n\}$ of integers by

$$b_1 = b_2 = b_3 = 1, \quad b_4 = 2, \quad b_n = 4b_{n-2} - b_{n-4} \qquad (n > 4).$$

Then

$$\begin{aligned} b_{n+1}b_{n-2} - b_nb_{n-2} &= (4b_{n-1} - b_{n-3})b_{n-2} - (4b_{n-2} - b_{n-4})b_{n-1} \\ &= b_{n-1}b_{n-4} - b_{n-2}b_{n-3}, \end{aligned}$$

so that, by induction, $b_{n+1}b_{n-2}-b_nb_{n-1}=1$, or $b_{n+1}=(1+b_nb_{n-1})/b_{n-2}$, and hence $\{a_n\}=\{b_n\}$.

II. *Solution by H. E. Bray, Rice University.* The solution of the problem is implicit in the following

THEOREM. *If $a_{n+1}=(k+a_na_{n-1})/a_{n-2}$ and $a_1=a_2=1$, $a_3=p$, where $k$, $p$ are positive integers such that $(k, p)=1$, a necessary and sufficient condition that $a_n$ be an integer is that $k=rp-1$, where $r$ is an integer.*

*Sufficiency.* Since $a_{n+1}a_{n-2}=k+a_na_{n-1}$ and $a_na_{n-3}=k+a_{n-1}a_{n-2}$, it follows that $(a_{n+1}+a_{n-1})/a_n=(a_{n-1}+a_{n-3})/a_{n-2}$, and this ratio is equal to $(a_3+a_1)/a_2=p+1$ if $n$ is even and to $(a_4+a_2)/a_3=(k+p+1)/p=r+1$ if $n$ is odd and $k=rp-1$. Thus $(a_{n+1}+a_{n-1})/a_n$ is an integer, and by recurrence $a_{n+1}$ is an integer if $a_1, a_2, \cdots, a_n$ are integers, whether $n$ is even or odd.

*Necessity.* We have $a_5=k+p(k+p)$, $a_6=[k+\{k+p\}\{k+p(k+p)\}]/p$, and since $a_6$ is of the form $(k+k^2+mp)/p$, where $m$ is an integer, it follows that $p$ divides $k(k+1)$. But if $p=1$, $k$ is of the form $rp-1$; and if $p>1$ then, since $(p, k)=1$, $p$ must divide $k+1$. That is, $k=rp-1$. This completes the proof.

Also solved by W. E. Barnes and P. A. Clement (jointly), W. J. Blundon, D. A. Breault, Brother Joseph Heisler, M. D. Burrow, Leonard Carlitz, W. J. Carpenter, Underwood Dudley, B. E. Fristedt and P. D. Rosenbaum (jointly), L. D. Goldstone, Virginia Hanly, Edward Harris, J. E. Homer, Jr. and David Zeitlin (jointly), A. R. Hyde, Erwin Just, William Kantor, Betty Levine, Y. L. Luke, Paul Manos, D. C. B. Marsh, Otto Mond, D. A. Moran, D. R. Morrison, R. F. Norris, H. O. Pollak, John Rainwater, David Rothman, William Ruckle, David Sachs, Mahmoud Sayrafiezadeh, Paul Schillo, Donna J. Seaman, George Senge, Arnold Singer, James Singer, Sister

Rita Jean Tauer, F. C. Smith, R. A. Spinelli, D. C. Stevens, Guy Torchinelli, Alan Wayne, Clement Winston, Dale Woods, and the proposers. Late solutions by A. C. Aitken, Vern Hoggatt, R. H. C. Newton, I. D. Ruggles, Norman Schaumberger, and Dmitri Thoro.

Pollak showed that the recurrence relation gives integers whenever $a_1$, $a_2$, $(a_3+a_1)/a_2$, $(a_4+a_2)/a_3$ are integers. Spinelli showed that, apart from translations, there are only two different positive integer sequences satisfying the recurrence relation, namely $a_1=a_2=a_3=1$ and $a_1=a_3=1$, $a_2=2$.

### Distinct Products in the Multiplication Table

E 1432 [1960, 802]. *Proposed by O. Lowenschuss and A. Rosenfeld, Budd Lewyt Electronics, Inc., Long Island City, N. Y.*

How many distinct products appear in the multiplication table through $N\times N$?

*Remarks by Paul Erdös, Israel Institute of Technology, Haifa, Israel.* The problem amounts to determining the number of integers not exceeding $N^2$ which can be written as the product of two integers not exceeding $N$. In a paper in the *Riveon Lematematika* in 1955 (in Hebrew) I proved that the number of these integers is $o(N^2/(\log N)^\alpha)$ for a certain $\alpha>0$. In a recent paper which appeared in the publications of the University of Leningrad (in Russian) I proved that if $A(N)$ denotes the number of these integers then for $N>N_0(\epsilon)$, $\epsilon>0$ arbitrary,

$$(N^2/\log N)(e\log 2)^{\log\log N/\log 2}(\log N)^{-\epsilon} < A(N)$$
$$< (N^2/\log N)(e\log 2)^{\log\log N/\log 2}(\log N)^{\epsilon}.$$

I do not see any way to obtain an asymptotic formula for $A(N)$.

### An Inequality for a Triangle

E 1433 [1960, 802]. *Proposed by Alexander Oppenheim, University of Malaya*

Let $P$ be a point in the interior of a triangle and let the distances of $P$ from the vertices of the triangle be $x$, $y$, $z$ and from the sides of the triangle be $p$, $q$, $r$. Show that $xyz\geqq(q+r)(r+p)(p+q)$.

*Solution by Leon Bankoff, Los Angeles, California.* In a triangle $ABC$ let $p$, $q$, $r$ denote the distances of $P$ from the sides $BC$, $CA$, $AB$ and let $x$, $y$, $z$ be its distances from the vertices $A$, $B$, $C$. In the circles described on the diameters $x$, $y$, $z$ we have

$$(q+r)/x \leqq 2\sin(A/2),\quad (r+p)/y \leqq 2\sin(B/2),\quad (p+q)/z \leqq 2\sin(C/2),$$

or $(q+r)(r+p)(p+q)/xyz\leqq 8\sin(A/2)\sin(B/2)\sin(C/2)$, with equality only when $p=q=r$. Since $8\sin(A/2)\sin(B/2)\sin(C/2)\leqq 1$, with equality only when $A=B=C$, it follows that $(q+r)(r+p)(p+q)\leqq xyz$, with equality only when $P$ is the incenter of an equilateral triangle.

Also solved by A. N. Aheart, Samuel Beatty, Leonard Carlitz, Ragnar Dybvik, L. D. Goldstone, Erwin Just, D. C. B. Marsh, L. J. Mordell, William Ruckle, Norman Schaumberger, D. C. Stevens, Dale Woods, and the proposer. Late solutions by A. C. Aitken, J. Basile, D. A. Breault, Robert Carlos, and M. V. Mielke.

*Editorial Note.* What is the analogue in 3-space of Oppenheim's inequality?

**Structure of Open Sets**

E 1434 [1960, 802]. *Proposed by Anatole Beck, University of Wisconsin*

Show that every open set in the plane can be represented as a disjoint union of closed straight line segments.

I. *Solution by D. A. Moran, University of Illinois.* Every open set in the plane is easily seen to be the disjoint union of open linear sets, *e.g.*, the intersections of the given open set with the set of all horizontal lines in the plane. But it is known that every open linear set is the disjoint union of open straight line segments. Thus the problem is reduced to showing that every open straight line segment can be represented as a disjoint union of closed straight line segments. But this is easily accomplished if we note, *e.g.*, that (0, 1) is the disjoint union of the closed sets $[1/3, 2/3]$, $[1/9, 2/9]$, $[7/9, 8/9]$, $\cdots$ (the union of all closed "middle thirds").

II. *Solution by J. C. Mathews, University of Oklahoma.* It is well known that every open set in the plane is a disjoint union of "half-open" rectangles of the form $\{(x, y): a \leqq x < b \text{ and } c \leqq y < d\}$. Thus if each of these rectangles can be shown to be a disjoint union of closed straight line segments, the problem is finished. To illustrate a general scheme that accomplishes this, consider the rectangle $R = \{(x, y): 0 \leqq x < 1 \text{ and } 0 \leqq y < 1\}$. First fill in the bottom half of $R$ as follows: (1) Let the vertical segment (abbreviated *VS*) $[0, 1/2]$ move on the horizontal segment (abbreviated *HS*) $[0, 1/2)$. (2) Let *HS* $[1/2, 3/4]$ move on *VS* $[0, 1/2]$. (3) Let *VS* $[0, 1/2]$ move on *HS* $(3/4, 7/8)$. (4) Let *HS* $[7/8, 15/16]$ move on *VS* $[0, 1/2]$. Etc. Next we note that the unfilled part of $R$ is again a half-open rectangle and a procedure like that above will fill it in. Etc.

III. *Solution by the Proposer.* A triangle with the entire base missing can be constructed as indicated in Figure 1. A triangle with two whole sides omitted is now made from this as indicated in Figure 2. Next we build a rectangle with two whole sides omitted as indicated in Figure 3. With such rectangles we can build any planar open set.

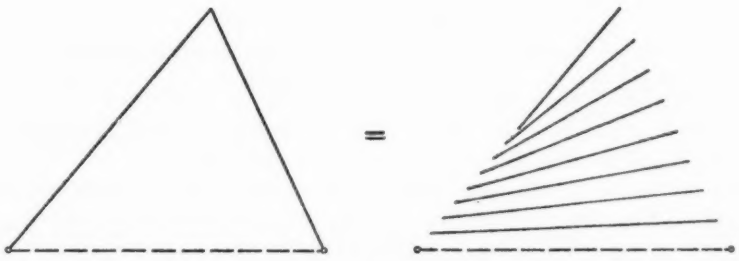

FIG. 1

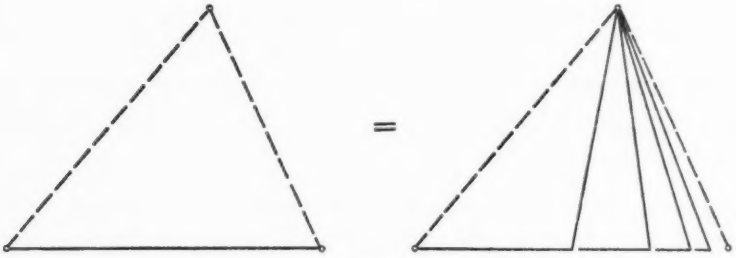

FIG. 2

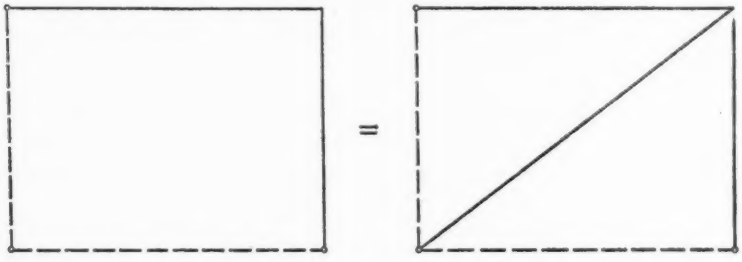

FIG. 3

Also solved, similarly to Solution II, by L. A. Ringenberg.

*Editorial Note.* Solution I is easily generalized to show that any open set in euclidean $n$-space can be represented as a disjoint union of closed straight line segments.

**Superior and Inferior Limits of Sequences of Sets**

E 1435 [1960, 802]. *Proposed by J. F. Leetch, Ohio State University*

Find the limit superior and the limit inferior for each of the following sequences of sets of real numbers:

(a) $\{A_n\}_{n=1}^{\infty}$ where $A_n$ is the set of integers mod $n$,

(b) $\{A_n\}_{n=1}^{\infty}$ where $A_n = \{a/n \mid a \in J(\text{integers})\}$.

*Solution by E. L. Cohen and Gerald Leibowitz, Massachusetts Institute of Technology.* (a) $A_n = \{0, 1, \cdots, n-1\}$, whence $A_1 \subset A_2 \subset A_3 \subset \cdots$ and $\limsup A_n = \liminf A_n = \bigcup_{n=1}^{\infty} A_n = \{0, 1, 2, \cdots\}$.

(b) Since $m/n = 2m/2n = 3m/3n = \cdots$, every rational belongs to infinitely many $A_n$. It follows that $\limsup A_n =$ set of all rationals.

Since $m = mn/n$, $J \subset A_n$. Let $m/n$ be a nonintegral rational, $n > 1$, $(m, n) = 1$. Then $m/n \notin A_{nk+1}$ for $k = 1, 2, 3, \cdots$ (since $m(nk+1)/n$ is not an integer). Thus there are infinitely many $A_i$ not containing $m/n$, and $m/n \notin \liminf A_n$. It follows that $\liminf A_n = J$.

Also solved by D. C. B. Marsh, D. A. Moran, D. C. Stevens, and the proposer.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4959. *Proposed by H. S. Shapiro and A. L. Shields, New York University*

(a) Exhibit an analytic Jordan curve $\Gamma$ in the complex plane (not a circle about the origin) such that $\int_\Gamma z^n ds = 0$ for $n = 2, 3, \cdots$, where $s$ is arc length.

(b) Exhibit an analytic arc $\Gamma$ (not a circle) such that $\int_\Gamma z^n ds = 0$ for all $n \geqq 1$.

4960. *Proposed by Donald L. Shell, Cincinnati, Ohio*

Prove for every positive integer $n$,

$$\sum_{p=1}^{n} (-1)^p \binom{n}{p} (p+1)^{p-1} p^{n-p} = (-1)^n.$$

4961. *Proposed by I. S. Gál, Yale University*

For $\sigma$, $t$ real, $\alpha(s) = \sum_{n=0}^{\infty} a_n n^{-s}$ and $a(u) = u^{-\sigma} \sum_{n \leq u} a_n$. Let

$$\hat{a}(t) = \int_0^\infty a(u) u^{-it} \frac{du}{u},$$

and define $\beta(s)$ and $b(v)$ similarly. Let

$$(a * b)(u) = \int_0^\infty a\left(\frac{u}{v}\right) b(v) \frac{dv}{v}.$$

Determine $\hat{a}$ and $a * b$ in terms of $\alpha$ and $\beta$.

4962. *Proposed by D. J. Newman, Yeshiva University*

Let $P(x, y)$ and $Q(x, y)$ be polynomials with real coefficients. Suppose $xP + yQ = 1$ for all points on the unit circle. Prove that $P = 0$, $Q = 0$ are satisfiable simultaneously.

4963. *Proposed by E. J. Burr, University of New England, N.S.W., Australia*

Let $k$, $r$, $n$ be three given integers such that $0 < k \leqq r \leqq n$. From a set of $n$ objects arranged in a line, $r$ objects may be selected in $\binom{n}{r}$ ways. How many of these selections have the property that at least one set of $k$ or more consecutive objects is included in the selection?

4964. *Proposed by Ernst Trost, Zürich, Switzerland, and Anders Bager, Hjørring, Denmark*

Consider a triangle *abc* divided into four smaller triangles, a central one *def* inscribed in *abc* and three others on the three sides of *def*. Show that *def* cannot have the smallest perimeter of the four unless all four perimeters are equal with *d*, *e*, and *f*, the midpoints of the sides of *abc*. (See also No. 4908 below.)

## SOLUTIONS

### Linear Combinations of Continuous Functions

4904 [1960, 382]. *Proposed by D. J. Newman, Yeshiva University*

Suppose a set, $S$, of functions continuous on $[0, 1]$ has the property that every linear combination of them has a zero on $[0, 1]$. Prove that there exists a nondecreasing function $\alpha(x)$ such that, for all $f \in S$, $\int_0^1 f(x)d\alpha(x) = 0$.

*Solution by I. J. Schoenberg, University of Pennsylvania.* For the case when the set $S$ is finite, the existence of $\alpha(x)$ is known (I. J. Schoenberg, *Convex domains and linear combinations of continuous functions*, Bull. Amer. Math. Soc., 1933, Thm. 2, p. 274. References to work of L. L. Dines are also given.) Let now the set $S$ be infinite. We associate to every $f \in S$ a set $M(f)$ of monotone functions defined by

$$M(f) = \left\{ \alpha(x) \,\middle|\, \int_0^1 f(x)d\alpha(x) = 0, \quad \alpha(0) = 0, \quad \alpha(1) = 1 \right\}.$$

This set is visibly nonvoid because $f(x)$ vanishes somewhere by assumption. The problem before us is to show that $\bigcap_{f \in S} M(f) \neq \varnothing$. By our remark concerning the case of finite $S$ a solution is directly implied by the following general

THEOREM. *Let $\{M\}$ be a collection of sets $M$ where each $M$ is a set of nondecreasing functions $\alpha(x)$, $0 \leqq x \leqq 1$, having the following properties:*

(i) *If $\alpha(x) \in M$ and the monotone $\beta(x)$ agrees with $\alpha(x)$ at $x = 0$, $x = 1$ and at all continuity points of $\alpha(x)$, then also $\beta(x) \in M$.*

(ii) *The elements of each $M$ are equi-bounded.*

(iii) *Each $M$ is closed with respect to point-wise convergence.*

*If for every finite subcollection of $\{M\}$ there is a common element, then there exists an element $\alpha(x)$ common to all the $M$.*

This will be seen to be a corollary of the following

THEOREM OF F. RIESZ. *If a collection $\{F\}$ of bounded and closed sets in $R^m$ has the property that the elements of every finite subcollection have a common point, then all the $F$'s have a common point.*

Riesz's theorem follows by a straightforward application of the Heine-Borel theorem. It was first published and proved by D. König [*Über konvexe Körper*, Math. Zeit., vol. 14 (1922), pp. 208–210]. To derive our theorem, let $n$ be a fixed

natural integer and let us denote by $\alpha^{(n)}$ the point of the $(2^n+1)$-dimensional space defined by

$$\alpha^{(n)} = (\alpha(0), \alpha(1/2^n), \alpha(2/2^n), \cdots, \alpha(1)).$$

The euclidean sets $F_m^{(n)} = \{\alpha^{(n)} | \alpha \in M\}$ are evidently bounded and closed by (ii) and (iii). Moreover, every finite number of them have a common point. By Riesz' theorem they all have a common point. This means: From each $M$ we can select a $\alpha_{n,M}(x)$ such that all these functions agree in value for $x=0, 1/2^n, \cdots, 1$.

Let $M_0$ be an arbitrary but fixed set of the collection $\{M\}$. Out of $\{\alpha_{n,M_0}(x)\}$ we select a subsequence such that for each $x$ in $[0, 1]$, $\lim_{\nu\to\infty} \alpha_{n_\nu,M_0}(x) = \beta(x)$. By (iii) we have $\beta(x) \in M_0$. Let now $M$ be an element of $\{M\}$. By construction we see that $\lim_{\nu\to\infty} \alpha_{n_\nu,M}(x) = \beta(x)$ for all binary $x = m/2^n$. By selection of an everywhere convergent subsequence of $\{\alpha_{n_\nu,M}(x)\}$ and by (i) we see that $\beta(x) \in M$. Thus $\beta(x)$ is common to all the $M$ and the theorem is established.

Also solved, using standard methods concerning linear operations in Banach space, by N. J. Fine, G. Lorentz, B. J. Pettis, Albert Wilansky, and the proposer.

**A Corollary of a Theorem of Schwartz**

4906 [1960, 479]. *Proposed by P. L. Butzer, Technical University, Aachen, Germany*

If the function $f(x)$ is continuous on an interval $(a, b)$ and, as $h\to 0$, $h^{-3}\int_0^h [f(x+u)+f(x-u)-2f(x)]du \to 0$ for all $x$ in $(a, b)$, then $f(x)$ is a linear function.

*Solution by Stephen Andrea, Olean, N. Y.* The numerator and denominator of

$$(1/h^3)\int_0^h [f(x+u)+f(x-u)-2f(x)]du$$

are both continuous differentiable functions of $h$. Hence we can apply l'Hospital's rule with the result

$$\lim_{h\to 0} (1/h^2)[f(x+h)+f(x-h)-2f(x)] = 0.$$

Schwartz's theorem now applies (Titchmarsh, *Theory of Functions*, 2nd ed., 13.84, p. 431) and the desired conclusion follows.

Also solved by Robert Breusch, C. Kassimatis, Y. Matsuoka, David Zeitlin, and the proposer.

**Subgroup of a Finite Abelian Group**

4907 [1960, 479]. *Proposed by P. T. Bateman, University of Illinois*

Suppose $\lambda_1, \cdots, \lambda_r$ are among the elements of a finite abelian group $G$ (written additively) and let $H$ be the subgroup of $G$ generated by $\lambda_1, \cdots, \lambda_r$. Show that there exist positive integers $k_1, \cdots, k_r$ such that every element of

$H$ is uniquely expressible in the form

$$n_1\lambda_1 + \cdots + n_r\lambda_r,$$

where $n_i$ is a nonnegative integer less than $k_i$ $(i=1, \cdots, r)$.

*Solution by N. J. Fine, Institute for Advanced Study.* Let $G_0=\{0\}$, $G_i=$the subgroup generated by $\lambda_1, \cdots, \lambda_i$ $(i=1, \cdots, r)$. Define $k_i$ as the least positive integer for which $k_i\lambda_i \in G_{i-1}$. Clearly every element of $H=G_r$ has a representation $\sum n_i\lambda_i$ with $0 \leqq n_i < k_i$. This yields a mapping from the set $N$ of $r$-tuples $(n_1, \cdots, n_r)$, $0 \leqq n_i \leqq k_i$, onto $G_r$. The number of elements in $N$ is $k_1 \cdots k_r$, and the number of elements in $G_r$ is

$$(G_r:G_{r-1})(G_{r-1}:G_{r-2}) \cdots (G_1:G_0) = k_r \cdots k_1.$$

Hence the mapping is one-to-one.

Also solved by Stephen Andrea, Robert Breusch, Joe Lipman, D. C. B. Marsh, J. G. Mauldon, D. A. Moran, D. T. Sandberg, Wu Ta-Sun, and the proposer.

**Partition of a Triangle**

4908 [1960, 479]. *Proposed by John Rainwater, University of Washington, Seattle*

Consider a triangle $abc$ divided into four smaller triangles, a central one $def$ inscribed in $abc$ and three others on the three sides of $def$. Show that $def$ cannot have the smallest area of the four unless all four are equal with $d$, $e$, and $f$ the midpoints of the sides of $abc$.

I. *Solution by P. H. Diananda, University of Malaya, Singapore.* Let $\alpha$, $\beta$, $\gamma (0<\alpha \leqq \beta \leqq \gamma)$ be the areas of the corner triangles and $\delta$ the area of the central triangle. Then we will prove that $\delta \geqq \sqrt{(\alpha\beta)}$ with equality if and only if $d$, $e$, $f$ are the midpoints of the sides of triangle $abc$. This result is slightly stronger than that proposed.

Let $bc$, $ca$, $ab$ be divided at $d$, $e$, $f$, respectively in the ratios $x:x'$, $y:y'$, $z:z'$ with $x+x'=y+y'=z+z'=1$. Also let $abc$ be of unit area. Then the corner triangles are of areas $y'z$, $z'x$, $x'y$. Also $\delta = 1-\sum y'z = xyz+x'y'z'$.

If $\gamma < \frac{1}{4}$ then $\alpha \leqq \beta \leqq \gamma$ and $\alpha+\beta+\gamma+\delta=1$ imply $\delta \geqq \sqrt{(\alpha\beta)}$. If $\gamma \geqq \frac{1}{4}$ then $\delta = xyz+x'y'z' \geqq 2\sqrt{(xyzx'y'z')} = 2\sqrt{(\alpha\beta\gamma)} \geqq \sqrt{(\alpha\beta)}$.

Equality is obtained if and only if $xyz=x'y'z'$ and $\gamma=\frac{1}{4}$, which are true if and only if $\alpha=\beta=\gamma=\delta=\frac{1}{4}$ and $x=y=z=\frac{1}{2}$.

II. *Solution by Anders Bager, Hjørring, Denmark.* Because the proposition is affinely invariant we may suppose $def$ equilateral. Let the notation be such that $\sphericalangle a \geqq \sphericalangle b \geqq \sphericalangle c$. Then $\sphericalangle a \geqq 60° = \sphericalangle dfe$, and the circular arc $fde$ reaches in the least as much "above" the common chord $fe$ as the arc $fae$ reaches "below." But $d$ is the "highest" point of the first mentioned arc. Hence we conclude that $\triangle def \geqq \triangle fea$, and the solution is complete.

The problem was proposed by H. Debrunner in *Elemente der Mathematik*, January 1956, and the solution here given appeared in the same journal in March 1957. The problem becomes much more difficult when each area is replaced by the corresponding perimeter. It is to this latter problem that P. Erdös gave some attention. It seems to have been Ernst Trost who called Erdös' attention to both problems.

Also solved by A. C. Aitken, Leon Bankhoff, W. J. Blundon, Robert Breusch, N. J. Fine, Michael Goldberg, Peter Yff, G. Laman, J. G. Mauldon, E. J. Mickle, Edmundo Morgantini (who published his solution as a note in *Rendiconti del Seminario Matematico dell' Universita di Padova*, XXX, 246–7), D. J. Newman, K. A. Post, G. E. Raynor, R. C. Read, G. B. Robison, I. J. Schoenberg, P. J. van Albada, J. H. van Lint, and the proposer.

**Zeros of a Special Polynomial**

4909 [1960, 479]. *Proposed by D. J. Newman, Yeshiva University*

Prove that the product of the zeros of $z^n \pm z^{n-1} \pm \cdots \pm z \pm 1$ which lie outside the unit circle is less than $\sqrt{(n+1)}$.

*Solution by Robert Breusch, Amherst College.* More generally:

If $f(x) \equiv z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ ($a_i$ real, $a_n = \pm 1$), then the product of the zeros of $f$ which lie outside the unit circle is less than $\sqrt{(1 + a_1^2 + \cdots + a_n^2)}$.

*Proof.* Let $f(z) = \prod_{k=1}^{n} (z - z_k)$, not $|z_k| = 1$ for all $k$, $|z_k| > 1$ for $k \leqq r$, $|z_k| \leqq 1$ for $k > r$. Let

$$f_1(z) \equiv \prod_{k=1}^{r} (z - z_k) = z^r + b_1 z^{r-1} + \cdots + b_r.$$

The $b_i$ are real because with any $z_k$ its conjugate is also a zero of $f_1$; and $b_r = (-1)^r \prod_{k=1}^{r} z_k$. Define further functions as follows:

$$f_2(z) = \prod_{k=r+1}^{n} (z - z_k) = z^{n-r} + \cdots \pm 1/b_r;$$

$$g_1(z) \equiv \prod_{k=1}^{r} (z - 1/z_k) = (-1)^r z^r \prod_{k=1}^{r} \left(\frac{1}{z} - z_k\right) \Big/ \prod_{k=1}^{r} z_k$$

$$= \frac{z^r}{b_r} f_1(1/z) = z^r + \cdots + 1/b_r;$$

$$g_2(z) = \prod_{k=r+1}^{n} (z - 1/z_k) = z^{n-r} + \cdots \pm b_r;$$

$$g(z) \equiv g_1(z) g_2(z) = z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \cdots + \frac{a_1}{a_n} z + \frac{1}{a_n};$$

$$A(z) \equiv f_1(z) g_2(z) = z^n + c_1 z^{n-1} + \cdots + c_n, \quad (c_n = \pm b_r^2);$$

$$B(z) \equiv f_2(z) g_1(z) = z^n + \frac{c_{n-1}}{c_n} z^{n-1} + \cdots + \frac{c_1}{c_n} z + \frac{1}{c_n}.$$

Thus $A(z) \cdot B(z) = f(z) \cdot g(z)$. The coefficients of $z^n$ in the two products give

$$(1 + c_1^2 + \cdots + c_n^2)/c_n = (1 + a_1^2 + \cdots + a_n^2)/a_n.$$

Thus $|c_n| = b_r^2 < (1 + \cdots + a_n^2)$, $|b_r| = \prod_{k=1}^{r} |z_k| < (1 + \cdots + a_n^2)^{1/2}$.

Also solved by T. V. Lakshminarasimhan, and the proposer.

## RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

*All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.*

*Differential Equations.* By Tomlinson Fort. Holt, Rinehart and Winston, New York, 1960. viii+184 pp. $4.75.

This is one of the better books available for an undergraduate course in differential equations. The level of mathematical rigor is satisfactory and the selection of topics is excellent.

The author's use of loosely defined differentials at times is at odds with the requirements of rigor; and the "symmetry in $x$ and $y$" which he achieves is illusory since he must frequently revert to derivatives in his discussion. Yet, common usage of the differential notation makes another completely consistent choice both difficult and awkward to maintain.

The careful selection of topics has both advantages and disadvantages. For the instructor who agrees with the author's choice it offers a consistent development of the material with little or no cutting either desirable or possible. However, for the instructor who disagrees, a compendium of methods and solutions gives him an opportunity to make some selection of topics as he cannot in such a book as Fort's. Furthermore, the author's pruning does result in a somewhat thin coverage, particularly in the latter part of the volume.

The book is clearly and carefully written and is most readable for a student. The approach is theoretical in nature; existence theorems and general solutions are both discussed; proofs are offered for most of the theorems. The physical applications have been sorted out into two chapters spotted through the book; one on "certain applications" and the other on "vibrations."

Throughout the book there is a nice balance between the desire for complete rigor and the teachability of an intuitive approach. The number of editorial and proofing errors is pleasantly small.

DONALD A. NORTON
University of California, Davis

*Combinatory Logic, Volume I.* By Haskell B. Curry and Robert Feys, with two sections by William Craig. North-Holland, Amsterdam, 1958. xvi+417 pp. $11.75.

Combinatory logic (here understood as including the intimately related theory of lambda-conversion) is a branch of mathematical logic concerned with the precise formal analysis of certain "prelogical" processes which are considered only informally in the usual current treatments; its primary aim is to provide for the "complete formalization" of logical systems. [Quoted phrases are from the book under review.]

This first volume of a projected two-volume treatise breaks naturally into two parts. Chapters 1–7 form a discussion of the fundamentals of the subject; in addition to bringing together a large number of known technical results from the theory of combinators (mostly due to Curry), this includes a treatment of lambda conversion and its relationship to the theory of combinators. With some interesting exceptions, the material in Chapters 1–7 is not new. In Chapters 8–10, the authors bring together much material from "illative" combinatory logic, which concerns the organization of objects into categories, especially as this is achieved using Curry's "functionality primitive". In large part, the material in Chapters 8–10 was previously unpublished.

This book is a valuable contribution to research in its field; as such it is not surprising that it is not entirely suitable as an introductory treatment. For the latter purpose, Chapter III, Section 4, of Rosenbloom's *Elements of Mathematical Logic* (1950) is probably the best source in English yet available. (The first part of Rosser's *Deux Esquisses de Logique* (1955) is even better.)

Many interesting questions are raised by this book. As presented, the theory aims (roughly speaking) at supplying underpinnings for the whole of mathematics (insofar as that is possible) in the same sense as does formal set theory (but more constructively). As developed, this involves formulation of an inherently combinatory predicate calculus; what are the details of such a formulation? Again: as developed, the underlying intuitive concept is taken to be function rather than set; precisely what system of set theory corresponds to the function theory realized in this book? Again: it is possible to formulate formal set theory (including the machinery of the classical predicate calculus) exactly as a combinatory logic. What are the most natural ways to achieve this formulation? And from the other side of the fence: what are the properties of combinatory logic as formulated within the framework of set theory? It is hoped that answers to some of these questions will be found in Volume II, along with material (promised by the authors) connecting combinatory logic with the theory of recursive functions.

DAVID E. SCHROER
University of Rochester

*Introduction to Linear Programming.* By Walter W. Garvin. McGraw-Hill, New York, 1960. xiv+281 pp. $8.75.

This is a particularly well-written account of the major topics in linear programming. Part I, entitled The General Linear Programming Problem, consists of five chapters which include a detailed description of the simplex method, sensitivity analysis (*i.e.*, the effect of changes in the coefficients on the optimal solution of a problem), and the gasoline-blending problem. Part II (five chapters), The Transportation Problem and Its Variants, includes chapters on unbalance and transshipment, assignment problems, a tanker-routing problem, and the generalized transportation problem. Part III on Special Methods (eight chapters) covers the following: upper bounds, statistical linear programming, revised simplex method, parametric linear programming with an application to a simple economic model, duality theory with application to the warehouse problem, and the resolution of degeneracy.

A clear and unhurried presentation, together with an abundance of numerical illustrations and flow diagrams, makes the book eminently suitable for self-study. The author's considerable industrial experience is evident in the well-chosen applications. The practitioner, teacher, or prospective student can all profit from a serious study of this volume.

H. KAUFMAN
McGill University

*Combinatorial Topology, Volume 3.* By P. S. Aleksandrov. Graylock Press, Albany, 1960. viii+144 pp. $6.50.

This volume, containing Parts Four and Five, apparently completes a translation of the first (1947) Russian edition of Aleksandrov's book. Part Four commences with (1) the duality theorems of Poincaré and of Alexander and (2) Pontryagin's theorems on linked systems of cycles. These results pertaining to homological manifolds are followed by an exposition of concepts needed for the topology of compacta and locally bicompact Hausdorff spaces. Part Four culminates in the Alexander-Pontryagin Duality and its consequences.

The final part is devoted to (1) the theory of continuous mappings of polyhedra, based ultimately on the work of Brouwer, and (2) the fixed-point theorem for topological polyhedra, as extended by Hopf from Lefschetz' fixed-point theorem for manifolds.

Aleksandrov's point of view is primarily geometric. His exposition is careful and detailed. The three volumes are a valuable compendium of fundamental concepts and results.

S. S. CAIRNS
University of Illinois

*Differential and Integral Calculus.* By J. R. F. Kent. Houghton Mifflin, Boston, 1960. xv+511 pp. $6.75.

A principal aim of the author was to write a text that a motivated student could understand without the aid of an interpreter and that is rigorous enough to answer some of his more searching questions. The reviewer feels that the author has constructed an interesting approximate solution to this difficult problem and that the text is worthy of critical perusal by those seeking a new calculus text.

Material is presented in the classical manner, that is, differential calculus followed by integral calculus. Exercises are carefully conceived and contain numerous thought-provoking questions as well as applications to the social and physical sciences. Definitions are given in the body of the text and are called out by italics. For the most part good heuristic explanations are given in lieu of proofs. Mean value theorems are given proper emphasis. The author's writing style and the publisher's printing format are both pleasing.

E. H. CRISLER
Hughes Aircraft Company

*Axiomatic Set Theory.* By Patrick Suppes. Van Nostrand, Princeton, N. J., 1960. xii+265 pp. $6.00.

Professor Suppes' book is an excellent, rigorous, clear text on axiomatic set theory for use in courses where students may not have any previous acquaintance with mathematical logic or set theory. Such a book has been needed for a long time. Students who work through this book and its many exercises should have no trouble with more advanced books on set theory, as for instance the *Axiomatic Set Theory* by P. Bernays and A. A. Fraenkel (North Holland Publishing Co., Amsterdam, 1958).

The Zermelo-Fraenkel system is developed here in detail. For didactic reasons, Professor Suppes adds an additional axiom (originally due to Tarski) when developing the theory of cardinals, but later he shows that the theory could have been developed without it. The book also covers the highlights of the theory of rational and real numbers, transfinite induction, ordinal arithmetic, well-ordered sets, the axiom of choice and its equivalents. There is a brief mention of the relative consistency of the axiom of choice on page 250, but some mention should be made there of the known results on the independence of the axiom (e.g., E. Mendelson, *The Axiom of Fundierung and the Axiom of Choice*, Arch. Math. Logik Grundlagenforsch., 1958, pp. 65–70, and E. Specker, *Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom)*, Z. Math. Logik Grundlagen Math., 1957, pp. 173–210.)

Now that such a lucid textbook is available it is hoped that its existence will encourage more colleges and universities to give courses on axiomatic set-theory.

L. N. GÁL
Yale University

*Naive Set Theory.* By Paul R. Halmos. Van Nostrand, Princeton, N. J., 1960. vii+104 pp. $3.50.

From the author's preface: "Every mathematician agrees that every mathematician must know some set theory; the disagreement begins in trying to decide how much is some. This book contains my answer to that question. The purpose of the book is to tell the beginning students of advanced mathematics the basic set-theoretic facts of life, and to do so with the minimum of philosophical discourse and logical formalism. The point of view throughout is that of a prospective mathematician anxious to study groups, or integrals, or manifolds. From this point of view the concepts and methods of this book are merely some of the standard mathematical tools; the expert specialist will find nothing new here. . . . Instead of *Naive Set Theory* a more honest title for the book would have been *An Outline of the Elements of Naive Set Theory.* "Elements" would warn the reader that not everything is here; "outline" would warn him that even what is here needs filling in . . . . The student's task in learning set theory is to steep himself in unfamiliar but essentially shallow generalities till they become so familiar that they can be used with almost no conscious effort. In other words, general set theory is pretty trivial stuff really, but, if you want to be a mathematician, you need some, and here it is; read it, absorb it, and forget it."

Having worked both in formal logic and in plain mathematics, Halmos is admirably qualified to write this book. It is possible that he was first prompted to set pen to paper when (as a Van Nostrand editor) he saw the manuscript of Suppes's *Axiomatic Set Theory.* In any event the two books form a natural contrasting pair, Suppes's for the careful logician investigating (say) the independence of the axiom of choice, Halmos's for the mathematician-in-the-street who just wants to stay out of trouble when he does (say) measure theory.

One small cavil. It seems to the reviewer that Halmos should have relaxed his "naive" principles occasionally, at least in his examples. A mathematician indifferent to most logical subtleties might nonetheless be delighted by the weird un-integer-like objects that happen to satisfy Peano's postulates.

H. Mirkil
Hanover, N. H.

*Ordinary Differential Equations and Their Solutions.* By George M. Murphy Van Nostrand, New York, 1960. ix+451 pp. $8.50.

This book is designed as a relatively exhaustive table of ordinary differential equations for which a solution, or a method of solution, is available. Considerable effort has been applied to provide a systematic classification of these differential equations, so that the locating of a particular equation is quite straightforward.

The text is composed of two parts. The first part contains a brief discussion of the methods available for solutions, and follows the classification scheme. The second part consists of over two thousand equations together with their solu-

tions, or appropriate references. Again, the cross-indexing makes it an easy matter to track down any particular differential equation.

The excellent organization of the material, and the ease with which it can be used, makes this work very valuable as a reference.

G. E. LATTA
Stanford University

*Elementary Statistics.* By Paul G. Hoel, Wiley, New York, 1960. vii+261 pp. $5.50.

Professor Hoel has written a text that will prove extremely useful for a one-semester service course in statistics for the student whose mathematical background has been limited to high-school algebra.

The topics considered include probability, theoretical frequency distributions, sampling, estimation, testing hypotheses, correlation, regression, chi-square distribution, nonparametric tests, analysis of variance, and time series and index numbers. The exercise lists are excellent and the tables necessary to solve them are included in the appendix.

Professor Hoel has a clarity of style and aptness of illustration that make each topic considered easy to understand and apply.

JOHN C. BRIXEY
The University of Oklahoma

## BRIEF MENTION

*High School Mathematics, Unit V, Relations and Functions.* University of Illinois Committee on School Mathematics. University of Illinois Press, Urbana, Illinois, 1960. 278 pp. Student's Edition $1.50, Teacher's Edition $3.00.

The fifth unit of UICSM Mathematics deals with relations and functions. The emphasis, of course, is on set theory. This admirable presentation and its many interesting exercises can well be considered by mathematicians everywhere.

*Digital Computer Fundamentals.* By Thomas C. Bartee. McGraw-Hill, New York, 1960. 334 pp., $6.50.

Some mathematics and some electrical circuit theory and a good bit of valuable information.

*Probability Theory.* (2nd ed.) By Michel Loéve. Van Nostrand, Princeton, N. J., 1960. 685+xvi pp., $14.75.

Two new chapters concerning random analysis have been added to Loéve's well-known text. A scholarly volume.

*Time Series Analysis.* By E. J. Hannan. Wiley, New York, 1960, 142+bib. pp., $3.50.

This book also assumes a reasonable statistical knowledge as a prerequisite, as well as some knowledge of infinite-dimensional vector spaces.

*Modern Factor Analysis*. By Harry H. Harman. University of Chicago Press, 1960 486 pp., $10.00.

If your library doesn't have a book on factor analysis available, this is one to be considered. With modern psychologists and sociologists becoming more and more interested in factor analysis, mathematicians will do well to have a reference book available. This one should help bridge the gap in vocabulary usage between mathematicians and psychologists.

*Mathematical Snapshots*, by H. Steinhaus. Oxford University Press, 1960. 318 pp., $6.75.

This revised and enlarged edition of Steinhaus's well-known pictorial work is a delightful book which should be called to the attention of high school teachers seeking suggestions for their libraries, as well as being welcome on college shelves.

*Introduction to Modern Algebra* (official textbook for Continental Classroom). By John L. Kelley. Van Nostrand, New York, 1960. ix+338 pp., $2.75.

By all means get up early and watch the program. One may wonder why modern algebra is introduced without an example of a postulational system such as a group, field, ring, etc., but upon reading the text, one finds that the basic heart of postulational algebra has not been slighted in the text, in spite of this. This reviewer's congratulations to the author and the committee for this courageous undertaking.

*Foundations of Geometry*. By Borsuk and Szmielew. North-Holland Pub. Co., 1960. 439+index pp., $12.00.

This is no gentle rehash of geometry designed for consumption by high school teachers at summer institutes, but a scholarly work on Euclidean and Bolyai-Lobachevskian geometry, as well as an introduction to projective geometry. Some excellent mathematical logic is displayed. This is a source book for serious geometers everywhere.

*The Number Story*. By H. Freitag and A. Freitag. National Council of Teachers of Mathematics, 1960. 76 pp., $0.85.

*Special Relativity*, by W. Rindler. Interscience, New York, 1960. 196+index pp., $2.25.

*Relativity, the General Theory*. By J. L. Synge. North-Holland Pub. Co., 1960. 414+app. and bib., $16.50.

This is a beautiful, mathematical presentation, which gets quickly to the heart of things without wasting the time of the specialist by attempting a lay presentation. Congratulations to Professor Synge.

*From An Ivory Tower*. By Bernard A. Hausmann, S. J. Bruce Pub. Co., Milwaukee, Wisconsin, 1960. 122 pp., $3.50.

A Jesuit scholar discusses certain philosophical problems of mathematical origin.

*Markov Chains with Stationary Transition Probabilities*. By Kai Lai Chung. Springer-Verlag, Berlin, 1960. ix+278 pp., DM 65.60. (About $16.25)

A well-written advanced volume in spite of the distracting translucency of the paper on which it is printed. A number of research problems still to be solved are mentioned at various points.

*Statistical Theory of Communication*. By Y. W. Lee. Wiley, New York, 1960. xviii+509 pp., $16.75.

Don't let the title mislead you. When Lee discusses the statistical theory of communication it is indeed a branch of higher mathematics, and rightly so. Probability theory and generalized harmonic analysis, as well as statistics and statistical mechanics, are used freely in this development of the Wiener-Lee theories.

*Stationary Processes and Prediction Theory.* By Harry Furstenberg, Princeton University Press, 1960, 283 pp., $5.00. Princeton Annals of Mathematics Studies, No. 44.

An advanced and carefully prepared volume on statistical predictability using Markoff and Weiner-Kolmogoroff methods.

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## NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.*

### PERSONAL ITEMS

Professor D. E. Christie, Bowdoin College, represented the Association at the inauguration of Dr. Kenneth Brooks as President of Gorham State Teachers College on December 7, 1960.

Professor George Dubay, University of St. Thomas, has been named "Piper Professor, 1960" and has received the Minnie Stevens Piper Foundation $1,000 award for outstanding scholarly and academic achievement.

Professor K. O. May, Carleton College, represented the Association at the inauguration of Dr. O. M. Wilson as President of the University of Minnesota on February 23, 1961.

*Kansas State Teachers College:* Mr. T. E. Bonner and Mr. D. L. Bruyr have been appointed Instructors; Assistant Professor L. E. Laird has been promoted to Associate Professor.

*Oregon State College:* Dr. C. S. Ballantine, University of California, Berkeley, has been appointed Instructor; Miss Florence A. Bakkum has been promoted to Assistant Professor; Professor I. M. Hostetter retired July 1, 1960.

Mr. S. O. Albert, Fort Bliss, Texas, has accepted a position as Retail Industry Representative in the Data Processing Division of International Business Machines, New York, New York.

Mr. C. D. Alders, Mankato State College, has been promoted to Assistant Professor.

Professor Howard Alexander, Earlham College, will be on leave during the year 1961–62 at Bowdoin College to lecture at a National Science Foundation Institute.

Mr. A. D. Brock, University of Oklahoma, has been appointed Assistant Professor at Radford College.

Miss Emalou Brumfield, Kent State University, has been appointed Teacher at Byron Junior High School, Shaker Heights, Ohio.

Mr. J. W. Calvert, University of Kentucky, has accepted a position as a member of the Technical Staff of Hughes Aircraft Corp., Fullerton, California.

Mr. C. J. Cillay, Texas State Board of Insurance, Austin, Texas, has accepted a position as Mathematician at the United States Naval Radiological Laboratories, San Francisco, California.

Mr. W. P. Durbin, Conviar, Fort Worth, Texas, has accepted a position as Engineer with the Radio Corporation of America, Moorestown, New Jersey.

Professor Wade Ellis, Oberlin College, has been awarded a Science Faculty Fellowship for 1961–62 by the National Science Foundation and will study at the University of Michigan.

Mr. L. C. Fletcher, Eastern Kentucky State College, has been appointed Head of the Department of Mathematics at Jackson Township High School, Circleville, Ohio.

Mr. H. S. Hall, Pennsylvania State University, has been appointed Assistant Professor at Rhode Island College.

Mr. G. G. Harrington, Jr., Boeing Airplane Company, Seattle, Washington, has accepted a position as Senior Engineer with the Martin Company, Denver, Colorado.

Mr. P. J. Hawkins, University of Connecticut, has been appointed Instructor in Electrical Engineering at Ohio State University.

Mr. M. O. Holoien, North Dakota State University, has accepted a position as Associate Research Engineer with Boeing Airplane Company, Seattle, Washington.

Mr. J. P. Lamb, University of Notre Dame, has accepted a position as Computer Programmer with International Business Machines, Washington, D. C.

Dr. E. O. Nelson, University of North Dakota, has been appointed Assistant Professor at the University of Utah.

Dr. P. B. Norman, Long Island University, has accepted a position as a member of the Technical Staff of Aerospace Corporation, El Segundo, California.

Mr. L. G. Salvin, Clark University, has accepted a position as Mathematical Statistician at the Cancer Chemotherapy National Service Center, National Institutes of Health, Silver Spring, Maryland.

Dr. E. T. Welmers, Bell Aircraft Corporation, Niagara Falls, New York, has accepted a position as a member of the Technical Staff of Aerospace Corporation, El Segundo, California.

Professor Emeritus W. E. Anderson, Miami University, died December 3, 1960. He was a Charter Member of the Association.

Professor L. S. Hill, Hunter College, died January 10, 1961. He was a member of the Association for 31 years.

Professor G. E. Moore, Eastern Michigan College, died June 20, 1960. He was a member of the Association for 36 years.

Mr. W. A. Riley, Jr., University of Kentucky, died July 15, 1960.

## SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1961:

*Cornell University*, June 29 to August 11: Mrs. Hertzig, survey of mathematics; Professor Walker, higher geometry.

*DePaul University*, Day, June 26 to August 4: Dr. De Cicco, lattice theory, infinite series; Mr. Czarnecki, introduction to Fourier series. Evening, June 12 to August 4: Dr. Yao, mathematical statistics.

*Indiana University*, June 14 to August 11: introduction to modern mathematics I; number theory; introduction to analysis I; introduction to topology I; non-euclidean geometry; mathematical reading and research.

*Kent State University*, June 19 to July 22: Professor Bush, selected topics for classroom teachers; Professor Dressler, differential equations I; Professor Jenkins, theory of numbers, history of mathematics; Professor Brooks, Boolean algebra. July 24 to August

26: Professor Johnson, college geometry; Professor Kaiser, differential equations II, functions of a complex variable.

*Northwestern University*, June 24 to August 5 and June 24 to August 19: advanced calculus; engineering mathematics I; numerical methods; statistics for teachers; statistics for experimenters; vectors, matrices, and quadratic forms; introduction to the theory of numbers; algebra for teachers; geometry for teachers; the history of mathematics II; advanced geometry for teachers; foundations of calculus for teachers; complex variables for applications; topics in modern mathematics for teachers; introduction to topology.

*Syracuse University*, July 5 to August 11: Professor Exner, mathematics of statistics, introduction to logic; Professor Baum, modern algebra; Professor Davis, axiomatic algebra in the "modern" school programs; Professor Hemmingsen, linear algebra; Professor Reid, real variables; Staff, digital computers and numerical analysis.

*University of Chicago*, June 19 to September 1: Dr. Fong, introduction to the theory of groups and rings; Professor Calderon, singular integrals and differential equations; Professor Stone, Hilbert space II; Professor Stinespring, introduction to functional analysis; Professor Baily, algebraic geometry. In addition to the above, special seminars in homological algebra and algebraic topology will be organized by Professors MacLane and Lashof and visiting mathematicians. Resident members of the Department of Mathematics will also make available to qualified students on an individual basis the customary courses entitled, "reading and research in mathematics."

*University of Michigan*, June 26 to August 19: Dr. Brumfiel, introduction to matrices; Dr. Clarke, theory of statistics I; Dr. Coburn, operational mathematics, Fourier series and applications; Mr. Cohn, differential equations, introduction to matrices; Dr. Craig, statistical analysis II; Dr. Dickson, advanced mathematics for engineers; Dr. Dushnik, operational methods for systems analysis; Dr. Dwyer, statistical analysis II, mathematical theory of probability I; Dr. Griffen, advanced calculus; Dr. Donald Jones, differential equations, mathematical theory of probability I; Dr. P. S. Jones, history of geometry and trigonometry; Dr. Kincaid, introduction to differential equations, operational mathematics; Dr. Lee, introduction to differential equations; Dr. Livingstone, topics in algebra, algebra; Dr. Nesbitt, mathematics of life insurance, calculus of finite differences; Dr. Reade, introduction to differential geometry, introduction to functions of a complex variable with applications; Dr. Rosen, theory of equations and determinants, set-theoretic topology; Dr. Schaefer, advanced mathematics for engineers; Dr. Ullman, introduction to functions of a complex variable with applications, real analysis I; Dr. Mrowka, introduction to the foundations of mathematics; Dr. Shimrat, Fourier series and applications, intermediate course in differential equations.

*University of Minnesota, College of Science, Literature and the Arts*, June 13 to July 15: Professor Harper, advanced algebraic theory; Dr. Joichi, advanced analytic geometry, theory of numbers; Dr. Miracle, differential equations, critical reasoning in mathematical analysis. July 18 to August 19: Professor Guggenheimer, non-euclidean geometry; Professor Harper, advanced algebraic theory; Professor Turner, critical reasoning in mathematical analysis, calculus of variations; Dr. Govindarajulu, probability.

*University of Oklahoma*, June 8 to August 5: Professor Bernhart, theory of games, college geometry; Professor Brixey, principles of mathematical statistics, theory of groups; Professor Huneke, theory of equations; Professor Springer, ordinary and partial differential equations.

*University of Pittsburgh*, June 5 to August 4: Professor Blumberg, advanced calculus; Professor Bowers, functions of a complex variable; Professor Bryson, partial differential equations and Fourier series; Professor Taylor, geometry of the complex domain; Professor Laush, integral equations. June 26 to August 4: Professor Knipp, differential equations; Professor Kovacs, mathematical theory of statistics; Professor Teats, history of mathematics; Professor Myers, introduction to modern algebra.

*University of South Carolina*, June 12 to August 12: Professor Lee, introduction to modern algebra; Professor Weber, college geometry.

*University of Tennessee*, June 12 to August 24: Professor Albert, Laplace and Fourier transform; Professors Harrold and Mahavier, seminar in topology.

*University of Utah*, June 26 to August 19: Dr. Nelson, integral equations; Dr. Barrett, topics in topology.

*University of Washington*, linear algebra; introduction to modern algebra; fundamental concepts of analysis; topics in applied analysis; foundations of geometry; non-euclidean geometry; special topics in mathematics; foundations of mathematics; Visiting Professor Mackey, special topics in analysis.

## BOOKS FOR ASIAN STUDENTS

The Books for Asian Students program of The Asia Foundation is now in its sixth year. Under this program over two million books and journals have been shipped to Asian educational institutions. The Asia Foundation will welcome donations of books suitable for the program. Suitable books are:

1. University, college, and secondary school books in good condition, published after 1945.
2. Scholarly, scientific, and technical journals in runs of five years or more.
3. Works by standard authors (e.g., Dickens, Hawthorne, Hemingway, Plato, W. James, T. Huxley, etc.)

The Asia Foundation will pay transportation costs from the donor to San Francisco and thence to Asia. Before shipping books to the Foundation, further information should be requested from: Books for Asian Students, 21 Drumm Street, San Francisco 11, California.

## EXPERIMENTS IN MATHEMATICS

The Board of Directors of the National Council of Teachers of Mathematics appointed a committee to gather information and to analyze experimental mathematics programs. This committee would like your help in identifying such programs. It has limited its analysis to programs that (1) have printed instructional materials other than courses of study, (2) are of at least one semester in duration, and (3) are not sponsored by a commercial publishing firm. The committee will examine all material, ask the experimenter to make a short personal report, and publish as many of the reports as possible.

If you know of experimental programs or have one of your own, we would appreciate receiving your material or having you contact the chairman, whose address is: Philip Peak, Chairman, Committee on the Analysis of Experimental Programs, School of Education, Indiana University, Bloomington, Indiana.

## AIR FORCE OFFICE OF SCIENTIFIC RESEARCH APPLIED MATHEMATICS PROGRAM

The Air Force Office of Scientific Research has announced a broad expansion of its support for research projects in the area of applied mathematics. This program which will emphasize both the traditional and modern aspects of applied mathematics will be a part of the program of the Directorate of Mathematical Sciences of AFOSR. It is the hope of the Air Force that this new program will help fill the increasing need for more mathematical methods that are required to solve today's complex problems in the physical and engineering sciences. Present plans call for the program to start in the fall of 1961. Interested applied mathematicians are encouraged to submit proposals for support under this program to the Director of Mathematical Sciences, Air Force Office of Scientific Research, Washington 25, D. C.

### MATHEMATICS AND THE WASHINGTON SCENE

An article under the above title by G. Baley Price, Executive Secretary, Conference Board of the Mathematical Sciences, appeared in the February, 1961, *Notices of the American Mathematical Society*. It is called to the attention of the readers of the MONTHLY since it was not possible to publish it in the MONTHLY.

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## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### THE NEW EDITOR-IN-CHIEF

The Board of Governors of the Association has elected Professor Frederick A. Ficken of New York University as Editor-in-Chief of the AMERICAN MATHEMATICAL MONTHLY for a five-year period beginning January 1, 1962.

Professor Ficken has served as Associate Editor of the MONTHLY from 1951 to 1956 in charge of *Mathematical Notes*. He has also served on the Committee on Slaught Memorial Papers from 1958 to 1960, and on the current Committee on Publications. He has been an MAA Secondary School Lecturer and a member of the Committee on Secondary School Lecturers from 1958 to the present time. He was Chairman of the Nominating Committee in 1960. He has wide interests and experience in both pure and applied mathematics.

After July 1, 1961, articles intended for publication in the MONTHLY should be sent to Professor Ficken at this address: Department of Mathematics, New York University, University Heights, New York 53, N. Y.

HENRY L. ALDER, *Secretary*

### THE FORTY-FOURTH ANNUAL MEETING OF THE ASSOCIATION

The forty-fourth Annual Meeting of the Mathematical Association of America was held at the Willard Hotel, Washington, D. C., from Wednesday to Friday, January 25 to 27, 1961, in conjunction with meetings of the American Mathematical Society, the Association for Symbolic Logic, and the Society for Industrial and Applied Mathematics. There were registered 1,527 persons, including 921 members of the Association.

Sessions of the Association were held on Wednesday and Thursday morning and on Friday morning and afternoon in the Grand Ballroom of the Willard Hotel. Presiding officers were Professor Rothwell Stephens on Wednesday morning, Second Vice-President Harley Flanders for the first lecture on Thursday morning, President C. B. Allendoerfer for the remainder of Thursday morning, First Vice-President A. S. Householder on Friday morning, and Professor B. J. Pettis on Friday afternoon. The Program Committee for the meeting consisted of E. E. Moise, Chairman; W. H. Durfee, Samuel Goldberg, S. A. Jennings, and B. J. Pettis.

### FIRST SESSION OF THE ASSOCIATION

#### Probability and Statistics

*Finite random walks*, by Professor Hale F. Trotter, Princeton University.

Suppose a particle takes successive unit steps in the lattice of points with integral coordinates in $k$-dimensional space, with each of the $2k$ possible directions equally likely at each step. Let $P(x, N)$ be the probability that a particle starting at $x$ visits the origin within the first $N$ steps. Then $\lim_{N\to\infty} P(x, N)=1$ when $k=2$ but not when $k=3$. (This was first proved by Pólya in 1921.) An elementary proof was given, showing that the different behavior in 2 and 3 dimensions reflects

the fact that the 2-dimensional (logarithmic) potential is unbounded at infinity while the corresponding 3-dimensional potential is not.

*Hitting probabilities*, by Professor Frank L. Spitzer, Princeton University.

The interplay between potential theory and Markov processes is used to study random walk on the lattice points of Euclidean space. It is shown how to calculate hitting probabilities (*viz.*, the probability that a finite set $A$ is first visited by the process $x_n$ at $a \in A$, given that $x_0 = b \notin A$). Such probabilities are represented in terms of the kernel $a(x, y) = \sum_{n=0}^{\infty} [P(x_n = x) - P(x_n = y)]$. The crucial theorem asserts the existence of the kernel for every pair of points $x$, $y$, with a positive probability of being visited, even in the recurrent case, when each series separately diverges.

*Applied statistical decision theory*, by Professor Howard Raiffa, Harvard University.

The problem is to choose an experiment $e$ from $E$ and after observing the experimental outcome $z$ in $Z$, to choose an act $a$ from $A$ to maximize the expected utility $E[u(e, z, a, \theta)]$, where $\theta \in \Theta$ is the unknown state parameter. It is assumed that to each $e$ in $E$ there is a given probability measure on $Z \times \Theta$. Special techniques are developed when (1) $u(e, z, a, \theta)$ is expressible in the form $u_a(e, z) + k_a + K_a w(\theta)$, (2) the conditional sampling measures on $Z$ given $e$ and $\theta$ are in the exponential family and (3) the conditional measures on $\Theta$ given $e$ and $z$ are conjugate to the sampling measures of (2).

## SECOND SESSION OF THE ASSOCIATION

*Metric entropy and approximation*, by Professor George G. Lorentz, Syracuse University.

Let $F$ be a compact subset of a linear metric space $X$. We wish to characterize the "massiveness" of $F$. For $\epsilon > 0$, let $n_\epsilon(F)$ be the minimal number of points in an $\epsilon$-net in $F$. Then $H_\epsilon(F) = \log n_\epsilon(F)$ is the entropy of $F$. An essential contribution to the 13th problem of Hilbert is possible with this tool. Let $d(F, L_n)$ be the supremum of the distances of points $x \in F$ to some $n$-dimensional subspace $L_n \subset X$. Then $d_n(F) = \inf_{L_n} d(F, L_n)$ is the $n$-dimensional width of $F$. For sets of functions, the exact values of $H_\epsilon(F)$ and $d_n(F)$ are usually unknown, but methods to determine their asymptotic behavior for $\epsilon \to 0$, $n \to \infty$ can be given.

*Annual Business Meeting of the Association*

*Undergraduate preparation for graduate work*, by Professor Andrew H. Gleason, Harvard University.

Analysis of even a strong program in mathematics at what is customarily regarded as undergraduate level reveals that only a small amount of mathematics is covered from a theoretical point of view. This is because it is impossible to present abstract mathematics to students until they have developed a certain *savoir faire*. Some students almost seem to be born with it; they can, and often do, start a graduate program in mathematics early in their undergraduate careers. Others acquire savoir faire slowly; still others never do. The speaker submits that the one thing that is truly important as preparation for graduate training in mathematics is the ability to think like a mathematician.

## THIRD SESSION OF THE ASSOCIATION

### Applied Mathematics

*Quasi-conformal mappings*, by Professor Lipman Bers, New York University.

This talk reviews the concept of quasi-conformal mappings and indicates its applications to applied mathematics, partial differential equations, and theory of functions, with emphasis on moduli of Riemann surfaces.

*Some problems in applied mathematics*, by Professor J. B. Keller, New York University.

Of all cylindrical rods with convex cross sections of unit area, which is the stiffest? This

optimum design problem in elasticity leads to the following mathematical problem in the calculus of variations. Which convex plane region of unit area yields the largest value for the minimum of the moment of inertia about any line through the centroid? The solution of this variational problem yields an isoperimetric inequality which gives an upper bound on the stiffness of any cylindrical rod with a convex cross section. Other optimum design problems in elasticity, heat conduction and other fields of application of mathematics lead to similar variational problems and isoperimetric inequalities.

*Applied mathematics as a science*, by Professor H. P. Greenspan, Massachusetts Institute of Technology.

There is, as yet, no general agreement concerning the nature of study and research in applied mathematics. As a direct consequence, this important area is badly neglected and does not appear in most university curricula as an independent entity or body of knowledge. Recognizing the need, a few major institutions, including the Massachusetts Institute of Technology, have developed comprehensive programs based on the philosophy that applied mathematics is primarily a science which seeks knowledge and understanding of physical phenomena through the use of mathematical methods. The emergent concept of the applied mathematician is that of a versatile scientist—a specialist in mathematics—with broad and active interests in many scientific areas, whose ultimate efforts are directed to the creation of ideas, concepts, and methods that are of basic and general applicability. The nature of applied mathematics is best explained by discussing its relationship to pure mathematics, theoretical physics, and engineering science. The distinctive attitude, approach and way of thinking is illustrated by several examples taken from current research in the fields of magneto-hydrodynamics and compressible gas dynamics.

## FOURTH SESSION OF THE ASSOCIATION

### Topology

*The topology of group-like spaces*, by Professor Eldon Dyer, University of Chicago.

As an example of the method of Algebraic Topology, a discussion is given of certain homology structures. The homology groups of cell-complexes are described. If the spaces in question have a continuous multiplication, their homology groups, with suitable coefficients, take on a richer structure, that of a Hopf algebra. The structure of such algebras is discussed and geometric applications are given.

*Some applications of homotopy theory to geometric problems*, by Professor Raoul Bott, Harvard University.

The notion of a smooth manifold is discussed with special emphasis on the parallelizability question.

*Some geometric applications of algebraic topology*, by Professor John W. Milnor, Princeton University.

An example is given of two finite simplicial complexes which are homeomorphic, but do not have isomorphic subdivisions. The proof depends on a recent theorem of B. Mazur. Let $L_i$ denote a 3-dimensional Lens manifold of type $(7, i)$; and let $K_i$ denote the space obtained from the Cartesian product of $L_i$ with a 5-cell by collapsing the boundary to a point. Then $K_i$ can be triangulated in a natural way. Using methods due to Reidemeister, Franz, and J. H. C. Whitehead, it is shown that no cell-subdivision of $K_1$ is isomorphic to a cell-subdivision of $K_2$. On the other hand, following Mazur, $K_1$ is homeomorphic to $K_2$.

## MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Wednesday afternoon in room 220 of the Willard Hotel in Washington with twenty-nine members present. Among the

items of business transacted were the following:

The Board approved the appointment by President Allendoerfer of the following Nominating Committee for 1961: R. P. Boas, Chairman; J. A. Clarkson, and C. R. Wylie, Jr.

The Board elected Professor Robert A. Rosenbaum of Wesleyan University as Second Vice-President for the two-year term 1961–1962.

The Board expressed its regret at the resignation of Professor Walter B. Carver as a member of the Finance Committee and approved the following resolution:

"Resolved that

The Board of Governors greatly regrets the resignation of Walter B. Carver as a member of the Finance Committee.

In accepting this resignation we wish to take note of his more than 40 years of service to the Association including terms as President, Secretary-Treasurer, and Editor of the MONTHLY, and to thank him deeply for all that he has contributed to the Association.

We wish him rapid recovery from his recent illness, and hope to see him frequently at future meetings of the Association."

The Board then elected Professor Carl B. Allendoerfer to fill the unexpired part of the term of Professor Carver ending in 1963 as a member of the Finance Committee.

The Board elected Professor Frederick A. Ficken of New York University as Editor-in-Chief of the MONTHLY for a five year period beginning January 1, 1962.

The Board acting upon the recommendations of its Committee on Publications under the Chairmanship of Professor R. P. Dilworth approved the following:

1. As soon as feasible, the volume year for the *Mathematics Magazine* should be changed to coincide with the calendar year.

2. The first editor of the *Mathematics Magazine* shall be appointed for a term extending through December 1963. Succeeding editors shall be appointed for five-year terms with the proviso that no editor may serve two consecutive terms.

3. Appropriate steps should be taken to insure that the transfer of the *Mathematics Magazine* to the Association be legally valid.

4. The Board of Governors, upon recommendation by the Editor, shall appoint the Editorial Board of the *Mathematics Magazine* for terms to coincide with the term of the Editor.

The Board then elected Professor Robert E. Horton of Los Angeles City College as Editor of the *Mathematics Magazine* for the three-year term January 1, 1961, to December 31, 1963.

The Board authorized continuation of the present subscription rates for the *Mathematics Magazine* with appropriate discounts to dealers and with a special subscription rate of $5.00 for 2 years available to members of the Association for subscriptions placed directly through the office of the Association.

The Board acting upon the recommendations of the Joint Committee on the Doctor of Arts Degree under the Chairmanship of Professor E. E. Moise voted that the degree of Doctor of Arts be established, in mathematics, at most of the universities which are qualified to grant the Ph.D. While the name of the degree was not considered part of the substance of the Committee's proposal, the Board approved a motion that the preferred name for the new degree be the Doctor of Arts. A more detailed report on the recommendations of this Committee will appear in the May issue.

The Board approved the following schedule of future meetings: Oklahoma State University, August 28–30, 1961; Sheraton-Gibson Hotel, Cincinnati, Ohio, January 1962; University of British Columbia, August 1962; University of California, Berkeley, January 1963; University of Colorado, August 1963; University of Michigan, August 1964;

Cornell University, August 1965; Rutgers-The State University, New Brunswick, New Jersey, August 1966.

The Board acting upon the recommendations of the Joint Committee on the Setting of Winter Meetings under the Chairmanship of Professor G. A. Hedlund approved:

1. that the practice of holding the Annual (Winter) Meeting near the end of January be continued,

2. that the representatives of the Society and Association appointed with the task of determining the place of the Annual Meeting should not preclude the possibility of holding the Annual Meeting in conjunction with a university,

3. that the determination of the time and place of such meetings be entrusted to a committee consisting of the Secretaries and Executive Directors of the stated organizations.

The Board authorized the Executive Committee to proceed with plans for celebration of the fiftieth anniversary of the Association which will occur in 1965.

## ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Thursday, January 26, 1961 in the Grand Ballroom of the Willard Hotel, Washington, D. C. with President Allendoerfer presiding. The Secretary reported that the membership of the Association was 10,197, an increase of 1,084 since the corresponding date last year.

The balloting for officers in which 2,032 votes were cast resulted in the election of Professor A. W. Tucker of Princeton University as President for the two-year term 1961–1962, and of Professors P. R. Halmos of the University of Chicago and John L. Kelley of the University of California, Berkeley, as Governors for the three-year term 1961–1963.

The Secretary then reported on some of the actions taken by the Board of Governors the previous day. He expressed, on behalf of the Association, the highest appreciation to all those responsible for the excellent arrangements for the meeting and singled out for special commendation Professor Everett Pitcher, the Chairman of the Arrangements Committee, Dr. Gordon L. Walker, Executive Director of the Society, Mrs. Robert Drew-Bear, for their most helpful efforts in coordinating the activities of the Society and the Association, and Professor John W. Brace for the latter's unusually effective efforts as Publicity Director for the meeting.

The two amendments to the By-laws which were printed on page 951 of the November 1960 issue of the MONTHLY were unanimously adopted.

President Allendoerfer then presented a summary of the status and activities of the Association.

## MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Monday, January 23, through Thursday, January 26. The thirty-fourth Josiah Willard Gibbs Lecture was delivered by Professor J. J. Stoker of New York University on Tuesday evening at 8:00 p.m. on "Problems in Non-linear Elasticity." At other sessions, Professor Lars Hörmander delivered an invited address entitled "On the Range of Differential Operators" and Professor Helmut Wielandt an address entitled "On the Structure of Finite Groups."

The Association for Symbolic Logic met on Tuesday. An invited address was given given at 2:00 p.m. by Professor S. C. Kleene on "Foundations of Intuitionistic Mathematics."

The Society for Industrial and Applied Mathematics met on Wednesday evening at 7:30 p.m. when Dr. Brockway McMillan delivered the retiring presidential address entitled "An Elementary Approach to the Theory of Information."

## ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of Everett Pitcher, Chairman; H. L. Alder, J. W. Brace, M. W. Oliphant, G. L. Walker.

Registration headquarters were located in the Caucus Room on the first floor of the Willard Hotel. The book exhibits were located in the Jackson Room on the first floor and the employment register in room 220. The Willard Hotel was the official hotel for the meeting, but the Washington Hotel (next door to the Willard) and the Raleigh Hotel (two blocks away) cooperated in reserving blocks of rooms.

HENRY L. ALDER, *Secretary*

## THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The nineteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at the City College of New York on April 2, 1960. Dr. B. G. Gallagher, President of the City College, gave the address of welcome. Professor J. P. Russell, Collegiate Vice-Chairman of the Section, presided at the morning session and Dr. George Grossman, High School Vice-Chairman, presided at the afternoon session. One hundred eighty-three persons, including 103 members of the Association, attended the meeting.

Professor Azelle B. Waltcher, Chairman of the Section, presided at the business meeting. The following proposal to amend the By-laws of the Section was approved: "The Executive Committee of the Section will include, in addition to the other members provided by the By-laws, one or more representatives of science and industry to be selected by the officers of the Section." Previously, the By-laws provided for only one such representative on the Executive Committee. Reports were presented by the Treasurer, Mr. Aaron Shapiro, by Professor J. N. Eastham for the Speaker's Bureau, and by Professor C. T. Salkind for the Committee on Contests and Awards.

The following papers were presented at the meeting:

1. *Nonstandard models of axiomatic theories*, by Professor Elliot Mendelson, Columbia University.

From the completeness theorem for first-order logic, it follows that axiomatic set theory has a model of every infinite power, and, in particular, a denumerable model, in apparent (but not real) contradiction of the fact that the existence of nondenumerable sets is provable in the theory. There are also nonstandard models in which the collection of ordinals of the model is not well-ordered by $\epsilon$. For any infinite power, there are at least $2^{\aleph_0}$ models of that power for formalized elementary number theory, and, even stronger, models having all elementary properties (including Goldbach's conjecture and Fermat's theorem or their negations) in common with the nonnegative integers.

2. *Some aspects of numerical analysis*, by Dr. H. H. Goldstine, International Business Machines, Yorktown Heights, New York.

The author first discussed the number system and arithmetical processes of digital calculation and showed the circumstances under which the familiar associative, distributive and commutative laws hold. Secondly, he discussed the topic of numerical stability and illustrated the concept with an analysis of the Bessel recurrence relations.

3. *The graph of a group*, by Professor Wilhelm Magnus, New York University.

The graph of a group was introduced by A. Cayley 100 years ago and utilized as an instrument of research by M. Dehn in 1910. It offers an access to group theory which has the advantage of giving a geometric (*i.e.*, visible) interpretation to groups. Also, the idea that a group element may be interpreted as a path provides an easy transition from group theory to some of its applications in topology, at least in an intuitive manner.

MARY P. DOLCIANI, *Secretary*

## THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The fifth annual meeting of the New Jersey Section of the Mathematical Association of America was held at Rutgers, The State University on November 5, 1960. Dr. H. O. Pollak, Chairman of the Section, presided at the morning session and Professor E. P. Starke presided at the afternoon session. There were 80 persons in attendance, including 59 members of the Section.

At the business meeting, an amendment to the By-Laws creating the office of Associate Secretary-Treasurer was approved. It was felt that this change would make it possible to divide the work of the office, and also provide a desirable continuity in the work. The Associate Secretary-Treasurer, elected for a three-year term, may succeed himself only once, but may subsequently serve as Secretary-Treasurer of the Section.

The following officers were elected: Chairman, Professor E. P. Starke, Rutgers, The State University; Senior Member of the Executive Committee, 1960–61, Dr. George Cherlin, Mutual Benefit Life Insurance Company; Member of the Executive Committee, 1960–62, Dr. Sheldon Meyers, Educational Testing Service; Member of the Executive Committee, 1960–63, Professor John Schumaker, Montclair State College; Associate Secretary-Treasurer, 1960–62, Mr. F. A. Varrichio, Saint Peter's College.

The report of Mr. R. S. Lockhart, Chairman of the Contest Committee, was presented by Dr. Cherlin, the Committee Secretary. 5,272 students from 139 schools participated in the 1960 contest, as contrasted with 4,462 students from 133 schools in 1959.

At the morning session the following papers were presented:

1. *Strategy in antimissile defense*, by Dr. W. T. Read, Jr., Bell Telephone Laboratories, Murray Hill, New Jersey (by invitation).

The application of mathematical analysis to problems lying in the broad, ill defined area between weapons system engineering on the one hand and military policy planning on the other was discussed. The approach was illustrated by using a simple model of anti-missile defense to derive various results relating to firing doctrine, deployment, offensive and defense strategies, and overall evaluation of defensive effectiveness.

2. *The rise of analysis through mechanics*, by Professor Solomon Bochner, Princeton University (by invitation).

Analysis rather than algebra constitutes the difference between Greek mathematics and "modern" mathematics which came into being during and after the Renaissance. Mathematics became the language of science mainly through analysis. Only after the great needs of classical mechanics and electrodynamics were satisfied could the present day algebraization of mathematics (and analysis) begin in earnest.

At the afternoon session, Professor B. E. Meserve, Montclair State College, reported on the work of the Panel on Teacher Training of the CUPM. There then followed a panel discussion:

3. *Careers for the mathematically gifted*, by Dean E. C. Easton, Rutgers, The State University (engineering), Dr. George Cherlin, Mutual Benefit Life Insurance Company (actuarial work), Professor E. R. Ott, Rutgers, The State University (statistics), Dr. H. O. Pollak, Bell Telephone Company Laboratories, Murray Hill, New Jersey (industry), Dr. Donald Thomson, International Business Machines Corporation, New York City (computing), and Professor B. E. Meserve, Montclair State College (teaching).

I. L. BATTIN, *Secretary*

## REPORT OF THE TREASURER FOR THE YEAR 1960

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1960. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of

the Association who wishes the complete report of the Treasurer may obtain it by writing to the Buffalo office of the Association.

The Current Fund ended the year with a surplus of $6,790 of which $5,000 has been transferred to the General Fund. Again this year contributions amounting to $204 have been added to the General Fund.

Among the temporary funds held by the Treasurer are certain items being held for the account of the Conference Board of the Mathematical Sciences. As soon as that Board becomes fully incorporated, these funds will be transferred to the Treasurer of the Conference Board.

| Assets of the Association | January 1, 1960 | December 31, 1960 |
|---|---|---|
| M. & T. Trust Co., Buffalo | $ 25,081 | $119,699 |
| Savings Accounts | 111,699 | 89,550 |
| Securities | 275,542 | 388,041 |
| | $412,322 | $597,290 |
| **Funds of the Association** | | |
| Current Fund | $ 2,526 | $ 4,316 |
| Carus Fund | 28,798 | 35,066 |
| Chace Fund | 9,546 | 9,062 |
| Houck Fund | 12,028 | 11,012 |
| Chauvenet Fund | 1,642 | 1,671 |
| Dunkel Fund | 19,745 | 21,410 |
| General Fund | 64,266 | 77,316 |
| | $138,551 | $159,853 |
| Visiting Lecturers | $ 78,423 | $ 79,580 |
| CUPM (Ford Grant) | 54,923 | 41,340 |
| CUPM (NSF Grant) | — | 164,809 |
| High School Contests | Deficit 77 | Deficit 5,828 |
| Survey Non-Teaching Math. Employment | 10,384 | 3,768 |
| Secondary School Lecturers | 73,883 | 52,091 |
| Production of Films | 15,437 | 8,711 |
| Washington Office | 34,154 | 48,732 |
| Survey European Math. Education | 6,644 | 3,980 |
| Institutes | — | 10,254 |
| Conference Board Survey | — | 30,000 |
| | $412,322 | $597,290 |

## OFFICERS AND COMMITTEES AS OF FEBRUARY 1, 1961

### OFFICERS

*President*, A. W. Tucker, Princeton University (1961–1962)
*First Vice-President*, A. S. Householder, Oak Ridge National Laboratory (1960–1961)
*Second Vice-President*, R. A. Rosenbaum, Wesleyan University (1961–1962)
*Editor*, R. D. James, University of British Columbia (1957–1961)
*Secretary*, H. L. Alder, University of California, Davis (1960–1964)
*Treasurer*, H. M. Gehman, University of Buffalo (1958–1962)
*Associate Secretary*, L. J. Montzingo, Jr., University of Buffalo (1958–1952)

## ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

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## CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.
ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.
INDIANA, Rose Polytechnic Institute, Terre Haute, May 6, 1961.
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PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.
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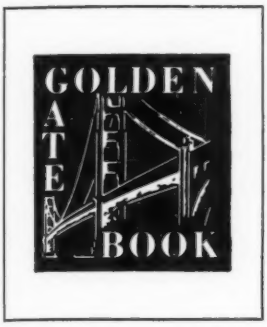


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